Everyday Drag

We start with a tale of viscosity. Drag is frictional in nature, so presumably we need a characterization of the fluid that gets at friction. We call this *viscosity*, which comes in two forms: *kinematic viscosity*, ν, in units $[Length]^2[Time]^{-1}$, and *dynamic viscosity*, $\mu = \nu\rho$, with units $[Mass][Length]^{-1}[Time]^{-1}$, or $\mathrm{Pa\cdot s}$ in SI. The dynamic viscosity is perhaps more intuitive, in that water "should be" more viscous than air. Indeed, water has a dynamic viscosity of 10^{-3} Pa·s at a temperature of 20 C (down by a factor of 3 at boiling temperature), while air has a dynamic viscosity of $2\times10^{-5}\,\mathrm{Pa\cdot s}$ at 20 C (double this at 160 C). Meanwhile, the kinematic viscosity is often more useful (in the Reynolds number, as we will see), and is similar to diffusion constant. The kinematic viscosity for water is about 10^{-6} m^2/s, and air is about 1.5×10^{-5} m^2/s (larger than for water!). Here is a table of various fluid viscosities.

Substance	density ($\mathrm{kg\cdot m^{-3}}$)	$\mu\,(\mathrm{Pa\cdot s})$	$\nu\,(\mathrm{m^2\cdot s^{-1}})$
air	1.3	2×10^{-5}	1.5×10^{-5}
water	1000	10^{-3}	10^{-6}
blood	1050	3×10^{-3}	3×10^{-6}
ethylene glycol	1100	1.6×10^{-2}	1.5×10^{-5}
olive oil	900	0.1	10^{-4}
corn syrup	1360	1.4	10^{-3}
peanut butter	1300	250	0.2

As we will later show, the diffusion constant for a medium, $D\sim\frac{1}{3}\lambda v$, where the mean free path, $\lambda\approx 1/n\sigma$. For air at STP, we have 6×10^{23} particles in 22.4 ℓ, or 0.0224 m^3 for a number density of $2.7\times10^{25}\,\mathrm{m^{-3}}$ and a cross section of approximately $\pi r^2\sim\pi(0.3\,\mathrm{nm})^2$, or $\sigma\approx3\times10^{-19}$ m^2 for a mean free path of about 100 nm. The thermal velocity is $v=\sqrt{\frac{3kT}{2m}}\approx 350$ m/s (about the sound speed), so $D\sim10^{-5}\,\mathrm{m^2\cdot s^{-1}}$, strikingly similar to our value for ν in the table above.

Buckingham Pi Approach

Let's use our new best friend to figure out how we might describe the force of drag on an object of characteristic radius R, mass m, at velocity v, in a fluid (water, air, syrup) of density ρ. We weill use kinematic viscosity, ν, to capture the frictional influence. Gravity may also be relevant, if looking at terminal velocity. So we make our table:

i	v_i	units	notes
1	F_{drag}	$\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}}$	sought quantity
2	R	m	object size scale
3	v	m/s	velocity
4	m	kg	mass (if we need)
5	ρ	kg/m^3	density of medium
6	ν	m^2/s	kinematic viscosity
7	g	m/s^2	gravity, if we need

We have $n=7$, $r=3$, suggesting four Π variables. We used kinematic viscosity instead of dynamic viscossity because the units are simpler and we have ρ in the problem already so can construct $\mu=\nu\rho$ should we need to. We'll go on a bit of a goose chase for illustrative purposes. So hang in there for a bit. We might start by constructing combinations that achieve a few dimensionless ratios in a somewhat obvious way:

$$\Pi_1=\frac{F_{\text{drag}}}{mg};\ \Pi_2=\frac{F_{\text{drag}}}{\rho R^2v^2};\ \Pi_3=\frac{Rv}{\nu}.$$

But what about the fourth one? It gets harder the more variable we have to construct independent combinations. For instance, we might note that $F_{\rm drag}/\rho\nu^2$ makes a fine pair, but it turns out that this is just $\Pi_2\Pi_3^2$, so nothing new. It may help to make a table of the powers of each variable in each Π construction:

i	v_i	Π_1	Π_2	Π_3	Π_4
1	$F_{\rm drag}$	1	1	0	
2	R	0	-2	1	
3	v	0	-2	1	
4	m	-1	0	0	
5	ρ	0	-1	0	
6	ν	0	0	-1	
7	g	-1	0	0	

To fill out the fourth column, we need something that is not a linear combination of the previous three. It's easiest to pick on something that does not appear often in other columns, like g. So how might we construct something with the same units as g without re-using m and $F_{\rm drag}$? How about v^2/R, for $\Pi_4 = v^2/Rg$? That works (and is known as the Froude number squared).

So why was this a goose chase? Because if we think more about it, drag force stems from the interaction of an object's geometry with the fluid, and should not depend on the mass, or gravity. These two things are relevant to terminal velocity, but once we have the drag force it is trivial to sort out the velocity under the condition that $F_{\rm drag} = mg$. Why bother with the goose chase, then? Because it illustrates both how to go about finding independent Π variables in a larger set, and also the roles that reason and intuition play in setting about things in a smart way.

So let's drop m and g from consideration. Now $n = 5$ and $r = 3$ so we have only two Π variables. The second and third variables grom before do not use m or g, so lets just recycle them:

$$\Pi_1 = \frac{F_{\rm drag}}{\rho R^2 v^2};\ \Pi_2 = \frac{Rv}{\nu},$$

and we set $\Pi_1 = f(\Pi_2)$ in the usual Buckingham fashion. Thus we have

$$F_{\rm drag} = \rho R^2 v^2 f\left(\frac{Rv}{\nu}\right). \tag{1}$$

If viscosity matters at all in the problem, then we expect drag to depend linearly on the viscosity (else re-define viscosity until it is useful in this proportional way). In order to achieve this, we need $f(x) = x^{-1}$, so that

$$F_{\rm drag} \approx \rho R^2 v^2 \left(\frac{Rv}{\nu}\right)^{-1} = \rho R v \nu.$$

That's tidy. It could be made even tidier by changing our viscosity variable/type by substituting $\mu = \rho\nu$. A formal soultion for a sphere finds that $F_{\rm drag} = 6\pi\rho R v \nu$ (known as Stokes' Law). Being off by a factor of 20 is not the most pleasing aspect of the Buckingham Pi approach, and is perhaps larger than would often be the case. But the essential physics is there.

In everyday life, we can understand drag as intercepting an oncoming flow and essentially robbing it of kinetic energy. A cross-section area A intercepts a volume $Av\Delta t$ in time Δt of air coming on at speed v. This parcel of air has kinetic energy $\frac{1}{2}\rho A v^3 \Delta t$, which constitutes a power $\frac{1}{2}\rho A v^3$. We relate this power expenditure to a force times velocity to get a drag force that looks like $F_{\rm drag} = \frac{1}{2}\rho A v^2$. Note that this looks like Eq. 1, but with no dependence on Π_2 (or we might say the zeroth-power of Π_2). This is an example of multiple valid solutions to a Buckingham solution, which formally was referenced by index, m in Lecture 5. So we have a second soultion to consider:

$$F_{\rm drag} \approx \rho R^2 v^2.$$

How do we know which one to use?

The Reynolds Number

The dimensionless parameter we formed in Π_2 above happens to be a famous and important characteristic of fluid flows called the Reynolds number, introduced by George Stokes and popularized by Osborne Reynolds.

$$\mathrm{Re} = \frac{Rv}{\nu}$$

The Reynolds number compares length scale and velocity to viscosity. A very small Reynolds number indicates that viscosity is important, while a very high Reynolds number points to a flow dominated by inertia rather than viscosity. The two regimes for drag we uncovered via Buckingham Pi are:

1. Viscous drag: $\mathrm{Re} \ll 1$; small things moving slowly
2. Inertial drag: $\mathrm{Re} > 100$, the drag is not a viscous phenomenon, but rather one of ram pressure

A crossover regime exists for Reynold's numbers of order 10. Let's look at some examples of Reynolds numbers to develop a feel. Using $\nu \approx 1.5 \times 10^{-5}\,\mathrm{m^2/s}$ for air and $\nu \approx 10^{-6}\,\mathrm{m^2/s}$ for water, we compute the following examples:

Action	R (m)	v (m/s)	ν (m^2/s)	Re
waving hand through air	0.1	5	1.5×10^{-5}	3×10^4
walking	0.5	2	1.5×10^{-5}	7×10^4
baseball pitch	0.05	40	1.5×10^{-5}	1.5×10^5
swimming	0.5	1	10^{-6}	5×10^5
car on freeway	1	30	1.5×10^{-5}	2×10^6
submarine at speed	4	10	10^{-6}	4×10^7
Boeing 747 at speed	4	300	1.5×10^{-5}	8×10^7

We do not personally experience viscous drag very often: only by watching tiny things in air/water do we tend to see this regime.

Viscous Drag Example

Let's do a real example: what is the terminal velocity of a marble in corn syrup? The marble is about 1 cm in diameter, and we expect its speed to be in the neighborhood of 0.1 m/s. So the Reynold's number is about $\mathrm{Re} \approx (0.01\,\mathrm{m}) \cdot (0.1\,\mathrm{m/s})/(10^{-3}\,\mathrm{m^2/s}) = 1$. Really, $\mathrm{Re} < 100$ is laminar, and viscous-dominated, so the marble in corn syrup should be in the viscous regime. Therefore, the drag force will be $F_\mathrm{d} \sim \rho\nu r v = \mu r v$. When this equals mg of the marble, or $\frac{4}{3}\rho_\mathrm{m} g \pi r^3$, terminal velocity is achieved. So $v \sim 4\rho_\mathrm{m} g r^2/\mu$, evaluating to 1.4 m/s ($r = 5$ mm; $\rho_\mathrm{m} \sim 2 \times 10^3$). Seems fast. We should do an experiment.

Stokes drag, when done full-up, carries a factor of 6π along with the μ. So we should divide our result by a factor of 20, to get 0.07 m/s. Not far from the initial guess, which honestly was *just* that, based on a mental picture of the (unperformed) experiment.

A dust grain in air, with diameter of about 20 μm ($R = 10\,\mu$m), will have a terminal velocity of around $v \approx \frac{1}{5}\frac{\rho_\mathrm{obj}}{\rho_\mathrm{air}} r^2 g/\nu$, or about 0.01 m/s if its density is about 1000 times that of air. This seems right for watching dust float around. For dust in air, then, $\mathrm{Re} \sim (0.01 \times 10^{-5}/1.5 \times 10^{-5}) \approx 0.01$.

Inertial Drag Done Right

We've seen, lived, and believed the scaling, also motivating it from a kinetic energy standpoint. Using thelatter form as a starting point, we can lump the remaining ignorance into a dimensionless drag coefficient, c_D, of order unity. Then we have:

$$F_\mathrm{drag} = \frac{1}{2} c_\mathrm{D} \rho A v^2. \tag{2}$$

The coefficient of drag goes as follows (from the Wikipedia gods):

Object	c_D	comments
Boeing 747	0.03	uses chord-times wingspan
sphere	0.1–0.4	uses frontal area; depends on smoothness
best cars	0.2–0.25	frontal area
pickup truck/SUV	0.5	frontal area
tractor trailer	0.8	frontal area
cube	0.8–1.0	depends on orientation to flow
man-bear-pig	1.0–1.4	most things fit here, if not built for streamline
flat plate	1.3	perpendicular to flow
Eiffel Tower	1.9	as with many French things

The range of c_D is not huge, within a factor of 2 of 0.5 for most things. Airplanes tend to use the chord of the airfoil times the wingspan (top area of wing) as the area, so the coefficient is not comparable directly to the others. Ships and submarines and swimming animals often use wetted area instead of frontal area, also lowering the number. For a trout, for instance, the wetted area $c_\mathrm{D} \sim 0.06$, while the frontal area $c_\mathrm{D} \sim 1.2$.

Since Eq. 1 led to two solutions, we can actually use Eq. 2 for everything, if we fold the viscous regime into c_D. In other words, we can treat $c_\mathrm{D} \sim \mathrm{Re}^{-1}$ for small Reynolds numbers. In fact, if we make $c_\mathrm{D} = 12/\mathrm{Re}$ for Reynolds numbers less than about 20, we nicely mesh into the inertial regime.

Gas Mileage

Let's consider the gas mileage for a pickup truck, with $c_\mathrm{D} \sim 0.5$, assuming all the energy goes into fighting drag. We use a frontal area of 4 m^2 (roughly a square of dimension 2 m), and a speed of 30 m/s to get a drag force of $0.5{\cdot}0.5 \cdot 1.3 \cdot 4 \cdot 900 \approx 1200$ N. The amount of work needed to go 1 mile (1.6 km) is then 1200 N$\times$1600 m, or about 2 MJ. Gasoline is about 10 kcal/g, so that 2 MJ (500 kcal) requires 50 g, or about 70 mℓ. But the combustion energy of the fuel is not delivered at 100% efficiency to the drive train. A typical efficiency would be 20% (about what you get from heat engine operating between 500 K and 350 K, realizing 50% of thermodynamic limit). So we need 0.35 ℓ to go one mile. Each liter will propel you about 3 miles, and with about 4 ℓ/gal, we get about 12 m.p.g. This is pretty close for a truck. Maybe too pessimistic, so 3 m^2 might be more realistic.

A car with half the drag coefficient and also half the projected frontal area will get four times the mileage, approaching 50 mpg.