Climate Change

An emerging problem of our age is the alteration of our planet's climate due to human modification of the atmospheric composition. Here, we explore the order-of-magnitude basis of the problem, and some of its implications.

Atmospheric Mass

First, what is the mass of the atmosphere? A surface pressure of 10^5 Pa dictates a mass of 10^4 kg over each square meter of the earth's surface. The earth's surface area, $A = 4\pi R_\oplus^2 \approx 12.5 \cdot 40 \times 10^{12}$ m^2, or $A \approx 5 \times 10^{14}$ m^2. This translates to a mass of $m = 5 \times 10^{18}$ kg. At 29 g/mol, we have 10^{44} air molecules.
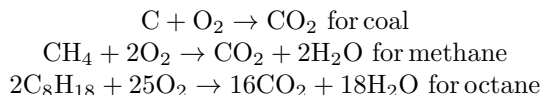
Carbon Dioxide

Pre-industrial levels of CO_2 contributed 280 ppm (parts per million) of the atmosphere by volume. Today, it is poking over 400 ppm, and climbing at a rate of about 2 ppm per year. Because CO_2 is 44 g/mol, its contribution by mass is greater—since the partial pressure of a gas is set by $P = nkT$, and n is the *volumetric* number density. The *mass* density is therefore proportional to nA, where A is the atomic mass number. This means that the fractional mass density of CO_2 in the atmosphere is $\frac{44}{29} \sim 1.5$ times the volumetric density.

Fossil Fuel Usage

We know from Lecture 2 that the world uses 15 TW of power, at least 80% of which derives from fossil fuels, or 12 TW of fossil fuel power. In a year, this translates into 4×10^{20} J of fossil fuel energy. At a typical energy content of 10 kcal/g, or about 4×10^7 J/kg, we must require 10^{13} kg of fossil fuel combustion per year.

Stoichiometry

If we look at the three fossil fuel reactions:

$$C + O_2 \rightarrow CO_2 \text{ for coal}$$
$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O \text{ for methane}$$
$$2C_8H_{18} + 25O_2 \rightarrow 16CO_2 + 18H_2O \text{ for octane}$$

we can deduce that every kg of coal combusted results in $44/12 = 3.7$ kg of CO_2; every kg of methane combusted results in $44/16 = 2.75$ kg of CO_2; and every kg of octane combusted results in $704/228 = 3.09$ kg of CO_2. So no matter what the source, **about 3 kg of CO_2 are produced for every 1 kg of fossil fuel input**. Note that coal is not only a bigger CO_2 contributor by gram of input, but coal has a lower energy content per gram. Meanwhile, natural gas has the highest energy content and the lowest CO_2 per gram of input. In net, coal is about a factor of two worse than natural gas in the more useful measure of CO_2 per unit of derived energy.
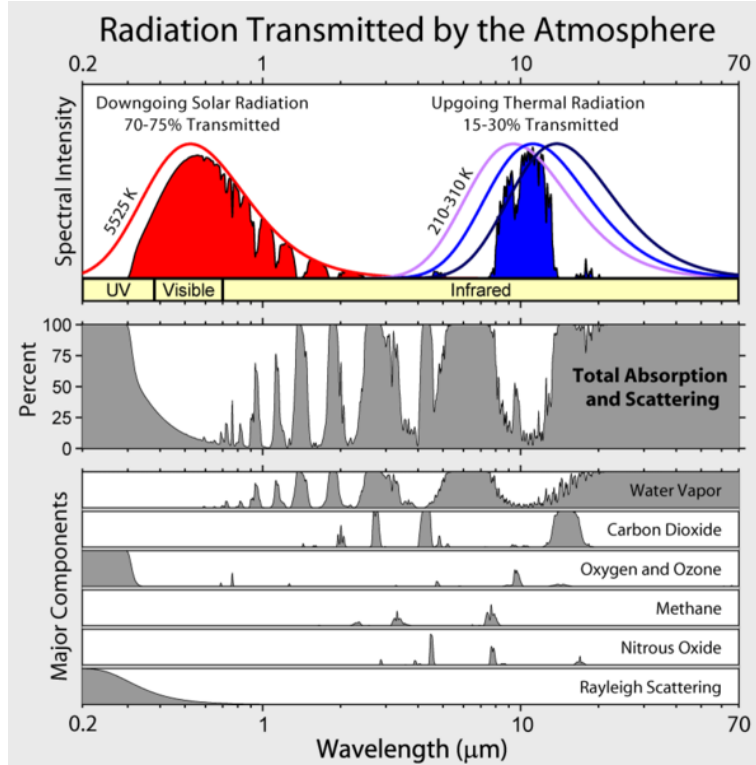
Figure 1: Key greenhouse gases. In the band where thermal emission takes place, water is almost the sole story, with CO_2 taking second place.

Net Carbon Dioxide

Since we require 10^{13} kg of fossil fuel per year, we produce about 3×10^{13} kg per year of CO_2, or about 4×10^{38} CO_2 molecules. One part per million of the atmospheric mass is 5×10^{12} kg, or 10^{38} molecules. So each year, we spew out 4 ppm by volume, or 6 ppm by mass. This is a little higher than the statement that the CO_2 trend rises by 2 ppm per year by volume. But at the very least, it shows that anthropogenic CO_2 is certainly *capable* of accounting for the trend. The discrepancy is because about half of the emitted CO_2 is absorbed by the ocean (making it more acidic) and does not show up in the atmospheric record.

Greenhouse Gases

The earth gets 1370 W/m^2 of incident solar light, but 30% of this reflects straight away. Meanwhile the flux incident on the projected πR^2 of Earth distributes across $4\pi R^2$ of surface area. The net effect is 240 W/m^2 on the planet's surface. Since in equilibrium all of this must re-radiate to space, the effective temperature—at unit emissivity—is $T_{\text{eff}} = 255$ K. Yet the average surface temperature on Earth is 15°C, or 288 K: 33 K higher than the radiated temperature of Earth.

The 33 K discrepancy is due to a thermal blanketing effect of "greenhouse gases." Some of the outgoing radiation is absorbed in the atmosphere and re-radiated: some up, and some back down. If we look at an infrared spectrum of absorbers in the average atmosphere (Figure 1), and ask how much of the 288 K ground radiation is absorbed, we find that about 70% is absorbed. Of this, about $\frac{2}{3}$ is from water, $\frac{1}{5}$ is from CO_2, and the remaining 15% is from other gases like methane.

Adding to the Greenhouse

Crudely speaking, if CO_2 is responsible for 7 of the 33 degrees of the greenhouse effect, we can easily predict the equilibrium consequences of an increase in CO_2, if the process is linear[1]. We have so far increased the concentration of CO_2 from 280 ppm to 400 ppm, or an increase of $\frac{10}{7}$, or 40%. I'm not sure if the 7 K contribution to the surface temperature is a current figure or a pre-industrial figure, so we'll look at it both ways and see it doesn't matter much at this level of analysis. If CO_2 increased the pre-industrial surface temperature by 7 K, then adding 40% more CO_2 would increase the temperature by ~ 2.8 K. If we instead say that 7 K is the current CO_2 contribution, the associated increase is ~ 2 K. Either way, the increase is in line with estimates of warming—though the system is slow to equilibrate due to the heat capacity of oceans, slowing down the rate of temperature increase. Another—perhaps more—critical factor is that much of the energy is going into melting ice, which absorbs energy without increasing temperature (into latent heat of fusion at 333 J/g melted). As less ice becomes available as an energy sponge, we may see more direct temperature surges. We have spent about half our total conventional petroleum, and less than half of our total fossil fuel deposits. Thus the ultimate temperature climb could be well over 5 K if we continue our practices unabated.

Modeling the IR Absorption

Single Layer Absorber Model

We can make a dirt-simple calculation based on the model that all the infrared action in the atmosphere is confined to a single thin layer. The global average input power is $P_{in} = 0.7 \cdot 1370\,\text{W/m}^2/4 = 240\,\text{W/m}^2$, and we assume all this is absorbed by the ground[2]. The upward radiation is then $P_{up} = \sigma T_s^4$—where T_s is the surface temperature. But some fraction, f, is absorbed in the atmosphere. Half of this is re-radiated upward, and lost to space, while the other half is directed back to the ground and absorbed. So the net radiation leaving the ground is $P_s = \sigma T_s^4 - \frac{1}{2} f \sigma T_s^4$. We might be tempted to say that the downward radiation is absorbed by the ground, then re-emitted, re-absorbed by atmosphere, with half leaving and half returning to the ground, and construct an infinite sum. But in our model, this would not make sense: the P_{up} is meant to represent the *total* upward radiation, given the equilibrium surface temperature, T_s, including the heating effects of radiation return from the atmospheric layer (what makes $T_s > 255$ K). From an energy balance, we must have $P_{in} = P_s$, so we are left with two unknown quantities: T_s and f. If we force T_s to be the observed 288 K, then we find $f \sim 0.77$, or 77% of the IR radiation is absorbed by the atmosphere.

In this model, if the single-layer atmosphere absorbed *all* IR radiation, we would end up with a surface temperature of 303 K, for a ΔT of 48 K (rather than 33 K observed).

Single Layer with Emissivity

We could also approach the problem from the point of view of emissivity of the single atmospheric layer, realizing that emissivity and absorptivity is the same, when assessed for the same (here thermal) spectrum. Instead of having non-absorbed radiation be *reflected*—as with a hunk of solid material—we will assume transmission. The radiation at the ground is then $P_s = P_{in} = \sigma T_s^4 - \epsilon \sigma T_a^4$, where T_a is the temperature of the atmospheric layer. We do not include a factor of $\frac{1}{2}$ for the atmospheric layer because both contributions assume emission into a half space, in the usual manner. For the atmospheric layer, we must have radiation balance, so that the absorption from ground radiation, $\epsilon \sigma T_s^4$ is balanced by the emission in both directions: $\epsilon \sigma T_s^4 = 2\epsilon \sigma T_a^4$, where the factor of two captures the bi-directional nature of the re-radiation. The last relation dictates that $T_s = 2^{\frac{1}{4}} T_a$. If we pick $T_a = 255$ K, then $T_s = 303$ K (too hot), and $\epsilon = 1$. If we set $T_s = 288$ K, we get $T_a = 242$ K (too cold), and $\epsilon = 0.77$. It is no coincidence that we get $\epsilon = f$ when we assign the surface temperature in this single-layer model, or that we get $\epsilon = 1$ when assigning the "correct" 255 K temperature for the atmosphere, since this is how the temperature of $T_{eff} = 255$ K is derived.

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[1] The process is not really linear, as the absorption lines are saturated. It's more logarithmic. But for small changes, we'll stay in the right ballpark.

[2] Some is actually absorbed in the atmosphere itself, but this is a crude model, so we'll roll with it.

Continuous Atmosphere

The atmosphere is obviously not a single layer. Each thin slice is capable of absorbing and re-emitting infrared radiation. If each vertical slice dz thick has probability pdz of absorbing infrared radiation, then the throughput from each slice will be reduced by $(1 - pdz)$. If we set up N steps covering the scale height, h, such that $dz = h/N$, the total throughput as the number of steps approaches infinity will be $(1 - ph/N)^N \to e^{-ph} = e^{-\tau}$, where $\tau \equiv ph$ is referred to as the *optical depth*.

For radiation traveling through a slice dz, the probability of being absorbed by the slice is pdz, which we can regard as the emissivity of the slice: $d\epsilon = pdz = \tau dz/h$. The portion of the upward bound radiation, P_{u}, absorbed by the slice is just $d\epsilon P_{\mathrm{u}}$. Meanwhile, the slice emits its own IR radiation at a rate $d\epsilon \sigma T^4$, where T is the height-dependent temperature. So we can write that

$$P_{\mathrm{u}}(z + dz) = P_{\mathrm{u}}(z) - d\epsilon P_{\mathrm{u}}(z) + d\epsilon \sigma T^4.$$

Likewise, we can say that the amount of downward radiation incident from above is absorbed in similar fashion, and the slice also emits its own downward radiation:

$$P_{\mathrm{d}}(z) = P_{\mathrm{d}}(z + dz) - d\epsilon P_{\mathrm{d}}(z + dz) + d\epsilon \sigma T^4,$$

which can be rearranged to give

$$P_{\mathrm{d}}(z + dz) = P_{\mathrm{d}}(z) + d\epsilon P_{\mathrm{d}}(z) - d\epsilon \sigma T^4,$$

ignoring higher orders in $d\epsilon$. We now have two coupled differential equations describing the power flow:

$$P_{\mathrm{u}}' = -pP_{\mathrm{u}} + p\sigma T^4$$
$$P_{\mathrm{d}}' = pP_{\mathrm{d}} - p\sigma T^4$$

Note that at any slice, the upward-going radiation must overpower the downward radiation by an amount equal to the power input (assume all visible light eludes greenhouse bands and is deposited on the surface). Otherwise heat would not be leaving the system. Thus $P_{\mathrm{u}} - P_{\mathrm{d}} = P_{\mathrm{in}}$. This implies that $(P_{\mathrm{u}} - P_{\mathrm{d}})' = 0$, which means $P_{\mathrm{u}} + P_{\mathrm{d}} = 2\sigma T^4$. At the top of the atmosphere, $P_{\mathrm{d}} = 0$, and $P_{\mathrm{u}} = P_{\mathrm{in}} = 240\,\mathrm{W/m^2}$. From this, we gather that the temperature at the top of our uniform-atmosphere is $T_{\mathrm{top}} = 214$ K. Given that the surface temperature is 288 K 8 km below, this translates to a lapse rate very close to the 10 K/km slope computed in Lecture 15. Note that T_{top} need not equal $T_{\mathrm{eff}} = 255$ K because this is no longer a unit-emissivity top layer.

We can throw this at a computer to crunch. We start with a surface temperature, $T_{\mathrm{s}} = 288$ K, thus $P_{\mathrm{u}}(0) = \sigma T_{\mathrm{s}}^4 = 390\,\mathrm{W/m^2}$, demand that $P_{\mathrm{d}} = P_{\mathrm{u}} - P_{\mathrm{in}}$, and at each step in height, adjust the temperature profile T such that we recover the conditional relation that $P_{\mathrm{u}} + P_{\mathrm{d}} = 2\sigma T^4$. Then we run the simulation for varying τ, and look for the condition that $P_{\mathrm{d}}(z = h) \to 0$. We can check the simulation to see that $P_{\mathrm{u}}(z = h) \to P_{\mathrm{in}}$, and that $T(z = h) \to T_{\mathrm{top}}$ as calculated above. Figure 2 shows the result.

I find that we need $\tau \sim 1.25$ to match conditions for Earth, which corresponds to a absorption probability of $1 - e^{-\tau} \approx 0.71$—not far from our simple-minded estimates of 77% absorption by a single layer. I verified that the results are not sensitive to the choice for h. This simulation does have one shortfall, however. The air is coupled to the ground only radiatively, which is a weak coupling for such a low-emissivity body. Thus the temperature of the air in immediate contact with the 288 K ground is a chilly 263 K. Convection must realistically play a role as well. If we force the air temperature at the surface to be 288 K, we require a very high $\tau \sim 2.25$, meaning 89% absorption, and a surface temperature of 308 K. I'm out of steam to include convection.

Runaway Greenhouse

If the atmosphere became choked with greenhouse gases—as is the case on Venus—then we would have an optical depth to infrared radiation $\tau \gg 1$. Here's what would happen to Earth (still assuming all the visible light radiation reaches the surface):
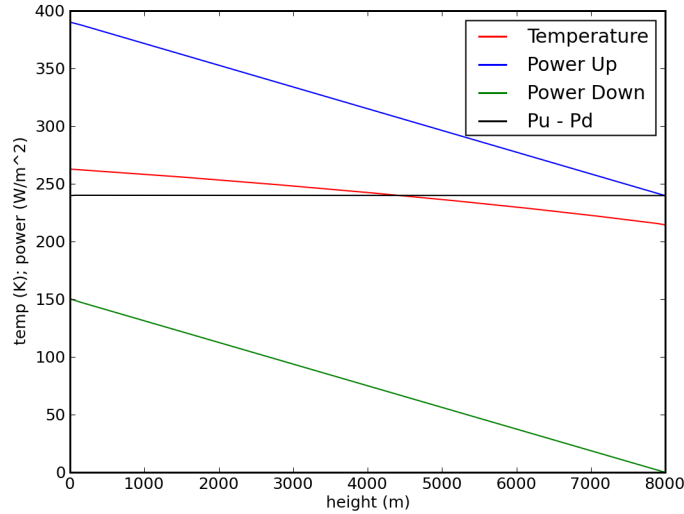
Figure 2: Simulation result. The difference between power up and power down is constrained to stay the same.

τ	T_s
1.254	288
2	303.5
3	321
5	349
10	400
100	684

Amusingly, given the high albedo on Venus (0.76), the net thermal flux from Venus into space is *less* than that for Earth, despite its being closer to the sun and having a surface temperature high enough to melt lead. With a semi-major axis of 0.72 AU, the incident solar flux is 2620 W/m^2, but the 24% that is absorbed means the average thermal emission is 157 W/m^2, for an effective temperature of 229 K—colder than Earth's 255 K! At a surface temperature of 735 K, I calculate $\tau \sim 210$, corresponding to an absorption length of something like 50 m.

Consequences of Warming

A few degrees of temperature increase may not sound like much, and may even seem a welcome change in some places. But the predicted consequences are many, and often complex to analyze. We will stick to a few simple consequences that we might easily calculate. In particular, we will look at sea level rise from two phenomena: thermal expansion of the ocean, and ice sheet melting.

Thermal Expansion

Water is at its most dense around 4°C. So its expansion is a bit tricky to evaluate, being roughly parabolic around 4°C. At 10°C, its expansion coefficient is 88×10^{-6}, becoming 207×10^{-6} at 20°C. Most ocean water is deep, and deep water is colder than surface water. It gets some geothermal input from the ocean floor, but let's assume we're dealing with water somewhere between 4–10°C and guess a thermal expansion coefficient around 20×10^{-6}. If we guess the average ocean depth to be 3 km, each degree of temperature rise in the ocean can be expected to raise its level by $3000 \cdot 20 \times 10^{-6} \sim 0.06$ m. So an ultimate 5 K warming would

result in 0.3 m of rise. Our conservatively small choice for expansion coefficient likely makes this number a lower bound.

Melting Ice

What about ice melt? For sea ice, there is no change, since the water is already displaced by the ice. Only land-borne ice may contribute. Looking at a globe, I would guess Greenland to occupy a few percent of the globe's surface area (let's say 2%, or 10^{13} m^2). If ice is piled up to an average of 1 km, then if distributed around the globe it would raise sea level by about 2% of this, or **20 m**. Sure, ice is not the same density as water, but nor do the oceans cover 100% of the earth area. So let's remain happy with our crude estimate. Is 1 km of ice reasonable over the $\sim$100,000 year lifetime of this ice sheet? We would only need to add 0.01 m per year, which might correspond to 0.1 m of fresh, fluffy snow. This does not seem at all unreasonable for a frigid environment where the air does not carry much water.

References

`http://www.aps.org/units/fps/newsletters/200807/hafemeister.cfm` is a good write-up on the physics of climate change. Many of the developments here parallel those of the article.

`http://geosci.uchicago.edu/~rtp1/papers/PhysTodayRT2011.pdf` is a Physics Today article on infrared radiative transfer in the atmosphere: a great introduction.