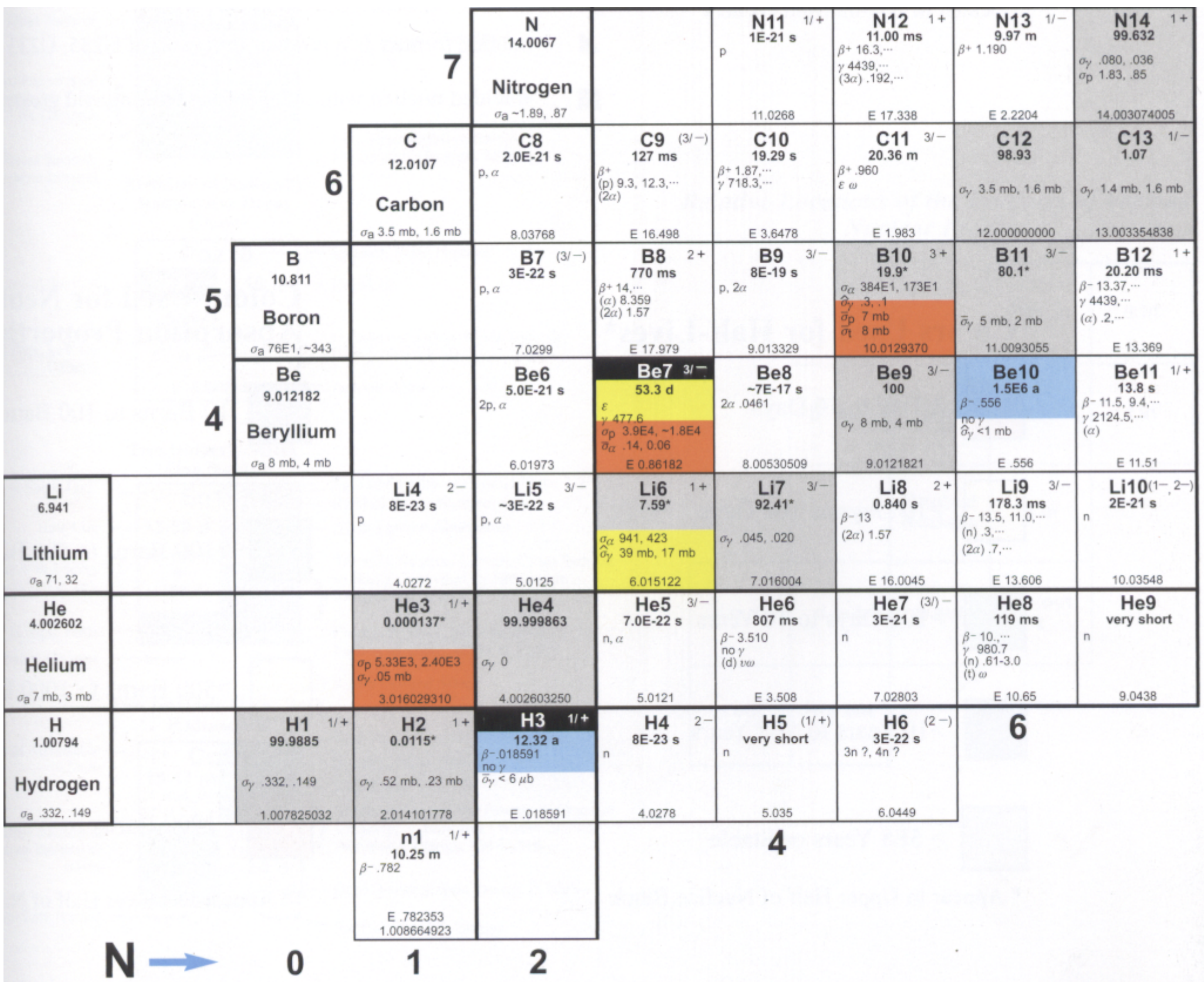
Nuclear Physics

What can a non-nuclear physicist tell you about nuclear physics that would be worth your while? Admittedly, I will only scratch the surface of the rich subject, and offer some quantitative tools and big-picture perspective.

The Physicist's Periodic Table

Hanging in virtually every classroom is the Chemist's Periodic Table of Elements. Child's play. Physicists need a Chart of the Nuclides. An interactive version can be found at `http://www.nndc.bnl.gov/chart/`. An excerpt at the bottom end appears below.



Proton number (Z) runs up the left side and neutron number (N) along the bottom. Ask a periodic table for the mass of a lithium atom and it'll say 6.941 a.m.u. The chart of the nuclides will say that lithium has two stable isotopes: ^{6}Li at mass 6.015122 a.m.u. comprising 7.59% of natural abundance, and ^{7}Li at 7.016004 a.m.u. making up the lion's share at 92.41%. Show off. You can also learn that ^{8}Li has a lifetime of 0.84 s, decaying by β^- to ^{8}Be, which itself lasts less than 10^{-16} s before breaking into two ^{4}He nuclei. And so it goes. The chart of the nuclides is a treasure trove of quantitative information about nuclei: masses, abundances, decay chains, half-lives, energy, neutron absorption cross section, excited states, nuclear spin. Really amazing.

You can see, for instance, that starting with four hydrogen atoms in the Sun (okay; nuclei, but their electrons are *somewhere* nearby) and ending up with a helium nucleus goes from 4 times 1.007825032 a.m.u. (4.03130013) to 4.00260325 a.m.u. for the helium. The mass difference is 0.0287 a.m.u., or about 0.71% of the initial mass. One atomic mass unit is 931.494 MeV, so we're talking 26.7 MeV. This translates to 4.3×10^{-12} J, and if we made a mole of helium, we'd get 2.6×10^{12} J per 4 g if material, corresponding to 150 million kcal/g. Thus we're talking about 15 million times more potent than chemical energy. So right from the chart, we can assess nuclear power potential.

Energy Sums

Let's get our numbers straight on what makes up the mass of an atom. We'll work from ^{12}C, which is the standard from which the a.m.u. is defined: ^{12}C is 12.000000 a.m.u. So what have we got? Six protons at 938.2720813 MeV apiece, plus six neutrons at 939.5654133 MeV each, plus six electrons at 0.510998 MeV all add to 11,270.090956 MeV. One a.m.u. is 931.494095 MeV, so we expect carbon to be 11,177.92914 MeV. The sum of the parts is therefore 92.1618 MeV *heavier* than the actual result.

The deficit is in *binding energy*, mostly in the form of nuclear binding, but some from Coulomb as well. It takes 1.03 keV to completely ionize carbon, so the "chemistry" piece accounts for a measly 0.001 MeV of the missing sum. The Coulomb repulsion from protons actually far outweighs the electron piece in the opposite direction, going approximately like $Z^2e^2/4\pi\epsilon_0 r$, where r is a few times 10^{-15} m (femtometer, or fermi) for nuclear dimensions. For carbon, this amounts to about 10 MeV.

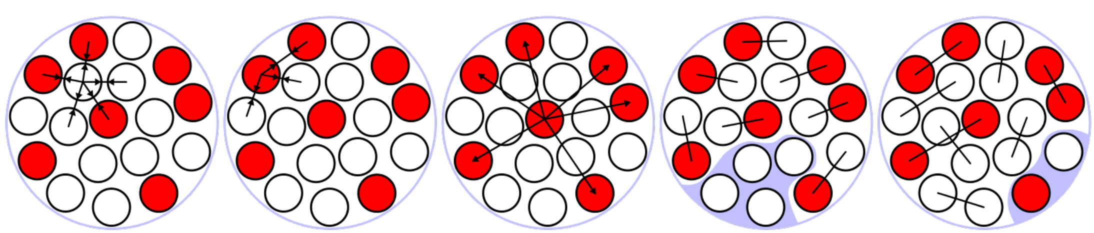
But the vast majority is from the **strong force**, coming in at close to 8 MeV per nucleon. Remember this figure: **8 MeV/nucleon**. That's the characteristic scale of nuclear binding energy.

One way to see this play out in numbers is that the proton and neutron average about 939 MeV, while the a.m.u. is about 931.5 a.m.u. The difference is about 8 MeV (per nucleon).

The Liquid Drop Model

Otherwise known as the semi-empirical mass formula (SEMF), Weizsäcker's formula, or the Bethe–Weizsäcker mass formula, the liquid drop model is a way to approximate the binding energy of a nuclide (thus the mass difference from the crude sum). It is composed of five terms, each having an intuitive/physical meaning (all units MeV). We first show the equation, then a graphic (thanks to the Wikipedia page), and finally a description of each term.

$$E_{\text{binding,MeV}} = 15.75 \cdot A - 17.8 \cdot A^{\frac{2}{3}} - 0.711\frac{Z^2}{A^{\frac{1}{3}}} - 23.7\frac{(A-2Z)^2}{A} + 11.18\frac{\delta_{\text{pair}}}{A^{\frac{1}{2}}} \tag{1}$$



1. Volume term: proportional to $A = Z + N$, adds 15.75 MeV per nucleon to account for nuclear binding force when completely surrounded by other nucleons. In the limit of insanely large nuclei (like neutron stars), the binding energy per nucleon would approach this value.

2. Surface term: proportional to surface area ($A^{\frac{2}{3}}$), subtracts 17.8 MeV per surface unit to account for the fact that surface neutrons are not surrounded. A little monkey business indicates a surface tension of 6×10^{17} N/m, putting water to shame! Incidentally, you can recover the surface tension for water by assuming a hydrogen bond strength of 0.44 eV per surface area of a sphere with radius ~ 3 Å (the scale one gets from the number density of water molecules).

3. Coulomb term: proportional to $Z^2/A^{\frac{1}{3}}$, reducing binding energy due to Coulomb repulsion. The scaling makes sense, as the potential energy from bringing in Z positive charges at characteristic scale R is $\frac{1}{2}Z(Z-1)e^2/4\pi\varepsilon_0 R$. The $Z(Z-1)$ term is reasonably approximated as Z^2 for large Z, and the linear scale of the nucleus, R, should go like $A^{\frac{1}{3}}$. In fact, if we make $R = r_0 A^{\frac{1}{3}}$, then we can tie the formula together to the energy scale (0.711 MeV) to find that $r_0 = 10^{-15}$ m, which is cute and tidy.

4. Asymmetric term: proportional to neutron excess squared: $(A-2Z)^2/A$, deducting from the binding energy when one species is out of balance with respect to the other. These extra neutrons (or protons, if it swings that way) must fill higher energy levels when unpaired with protons. This cost competes with Coulomb repulsion to determine how many neutrons vs. protons a nucleus will host: we have a cost for too many protons (Z^2) and a cost for too many excess neutrons (ΔN^2). Together, these create a valley of stability (lowest energy; highest binding energy).

5. Pairing term: the spins of these fermions find lower energy states (tighter binding) when there are an even number of protons and an even number of neutrons. If *both* Z and N are even, we add $11.18 \cdot A^{-\frac{1}{2}}$. If even–odd, no net energy is added or subtracted. If both are odd, we *subtract* $11.18 \cdot A^{-\frac{1}{2}}$.

Example Nuclides

Let's look at the carbon example again: $Z = 6$, $N = 6$, $A = 12$. The volume term starts us off with 189.0 MeV. The surface term subtracts 93.3 MeV leaving 95.7. The Coulomb term subtracts 11.2 MeV to make 84.5. We have no neutron excess, so nothing from that. Our even-even scenario adds a final 3.2 MeV for a total binding energy of 87.8 MeV. A computation of the mass of a neutral carbon atom would then look like $6 \times 938.272 + 6 \times 939.565 + 6 \times 0.511$ to get protons, neutrons, electrons, minus 87.8 MeV for the net nuclear binding energy for a total of 11,182 MeV or 12.005 a.m.u. Not bad (0.05% on the whole), but the binding energy piece is off by about 5%. Yet the low-A end is not where this thing shines (at low-A, the nuclide is lumpy and not as well characterized as a sphere).

For ^{56}Fe ($Z = 26$; $N = 30$), we calculate 55.937, compared to 55.935 (0.003%), and the binding energy (490.6 MeV) is good to about 0.3%. Let's see if it meets the gold standard: ^{197}Au ($Z = 79$; $N = 118$) gets us 196.972 (truth 196.9666) and binding energy about 1554.5 MeV, good to 0.3%. Happy?

Binding Energy per Nucleon

We noted before that the binding energy comes out to about 8 MeV per nucleon. Although Eq. 1 starts out strong with 15.75 MeV per nucleon, it gets chipped away by surface terms, Coulomb repulsion, and asymmetry as neutrons pile up. The net result is indeed about 8 MeV per nucleon, but this is not a constant. Figure 1 shows the prediction of the model on top of real data. Not a bad fit—especially for high mass number.

You are probably already familiar with the significance of this curve, but I can't go this far without commenting. The curve has a peak, and this occurs around $A = 56$, where iron sits. It means that ^{56}Fe is the most tightly bound nucleus. Beyond iron, Coulomb repulsion starts weakening the nucleus. Fusion in stars finds energy gain in climbing the left side of the curve, but is energetically unable to transcend iron, at the peak. Fission delivers energy by climbing toward the peak along the gentle slope on the right (but the energy gain is less dramatic than is the case in fusion.

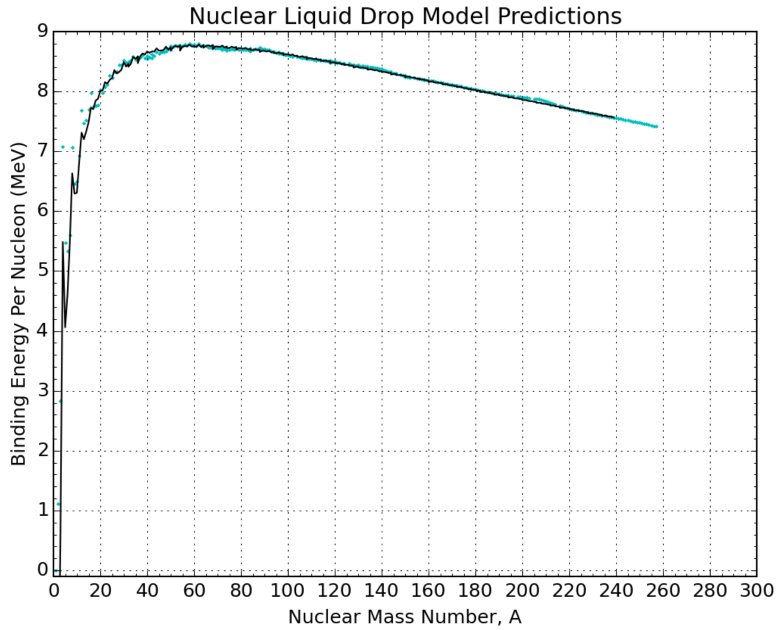


Figure 1: Binding energy per nucleon. Real data in blue points, and the model as a black line. It struggles for low mass number, but really finds its grove later in life.

Valley of Stability

How do we know which nuclei are stable? The binding energy tells much of the story—at least with respect to beta decays. For a given nuclear mass, A, the first two terms in the liquid drop formula are unchanged: vanilla nucleons attracted similarly by the strong force. For now, let's consider an odd value for A, which guarantees an even–odd or odd–even count of protons and neutrons, so that the pairing term is net zero. Then as we take a diagonal slice through the chart of the nuclides (constant A), we need only consider the Coulomb and asymmetry terms. The stable nucleus will be the one with the highest binding energy, which means that we want to minimize the sum of these terms:

$$\text{minimize}\left(0.711\frac{Z^2}{A^{\frac{1}{3}}}+23.7\frac{(A-2Z)^2}{A}\right). \tag{2}$$

Taking the derivative with respect to Z and setting to zero leads to the solution

$$Z=\frac{A}{2+\frac{1}{2}\frac{0.711}{23.7}A^{\frac{2}{3}}}.$$

The numerical factor is just the ratio of coefficients for the Coulomb and Assymetry terms in the model. Because this is a small number, when A is small enough, the $A^{\frac{2}{3}}$ term is negligible and we have $Z=A/2$, meaning equal neutron and proton numbers. We are familiar with the fact that up through about ^{40}Ca ($Z=N=20$), we tend to see comparable numbers of neutrons and protons in a nucleus (at least the most abundant isotopes). At this stage, the denominator has moved from 2.0 to 2.175, and the expectation for Z has dropped to 18.5. By the time we get to ^{208}Pb ($Z=82$, $N=126$), the formula suggests $Z=82$ (it works!). The chart of stable nuclei therefore rolls over from the $Z=N$ line toward neutron-rich territory, for the intuitive reason that protons are repulsive to each other. Figure 2 displays the results of seeking local maxima in the binding energy for each value of A.

For a fixed value of A, traveling far from the minimum Z value results in a higher energy. In fact, we see in Eq. 2 that it is quadratic in nature. Thus an $A=$ const. diagonal slice across Figure 2 results in an energy

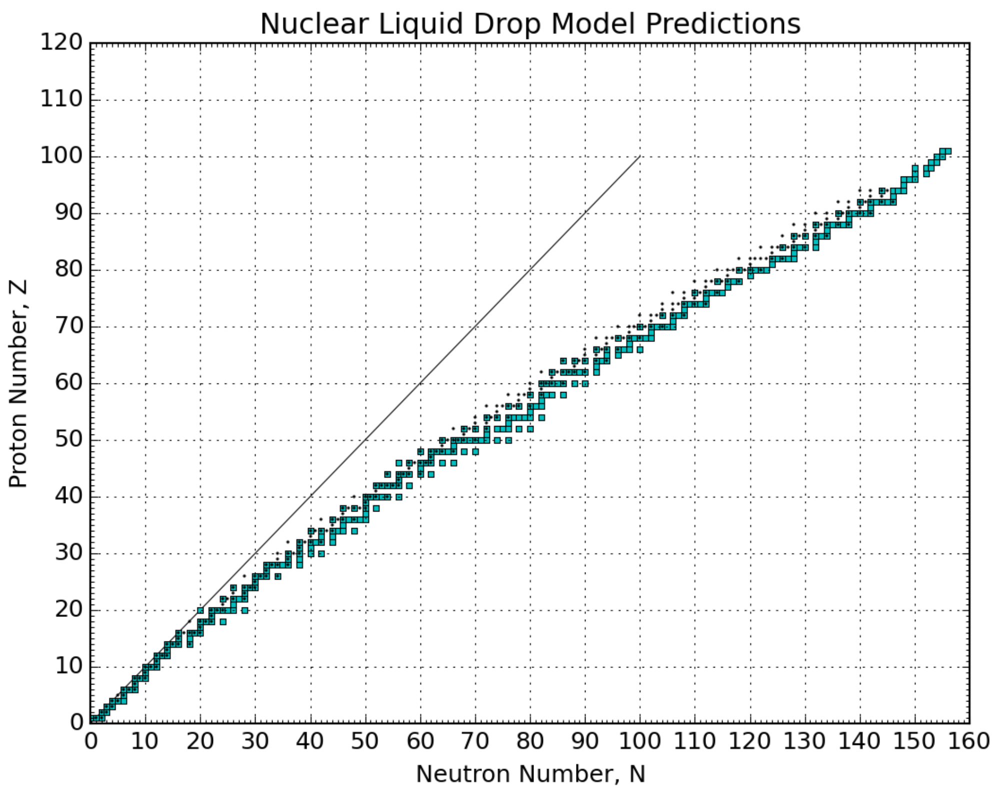


Figure 2: Local maxima in binding energy for each nuclear mass. Blue squares are from real data, while black dots are predicted from the amazingly simple liquid drop model.

valley such that the stable nuclei lie at the bottom (maximizing binding energy results in minimum nucleus energy, since binding energy is subtracted from the total). This is referred to as the Valley of Stability.

The predictions in Figure 2 seek local maxima in binding energy in a way analogous to the derivative method above, except that the pairing term can add structure to the curve for even values of A. As noted before, odd values of A guarantee even–odd pairings, contributing nothing to this term. But traveling along constant A for even values of A results in oscillating between even–even and odd–odd sets, so that the binding energy oscillates. The valley floor is not smooth, then. When the slope is small, pairing oscillation can make multiple local minima. Once on the slope, no such luck. This is why some values of A produce multiple stable isotopes (for even A)—giving the chart a fragmented appearance. At low-A, the valley walls are immediately too steep to allow these shenanigans, but later on they take off.

One thing the liquid drop model does *not* get is *magic numbers*. When *either* Z or N are 8, 20, 28, 50, 82, or 126, we tend to see additional stable nuclei that the liquid drop model would not predict. Tin, at $Z = 50$, has a whopping 10 (pun there) stable isotopes. Some isotopes are doubly magic: ^{16}O, ^{40}Ca, ^{48}Ca, and ^{208}Pb.

Summary

The liquid drop model is not perfect, and by no means a replacement for the incredible Chart of the Nuclides. But it's pretty amazing that it does so well, and that each piece carries an intuitive meaning that helps us understand the nucleus better.