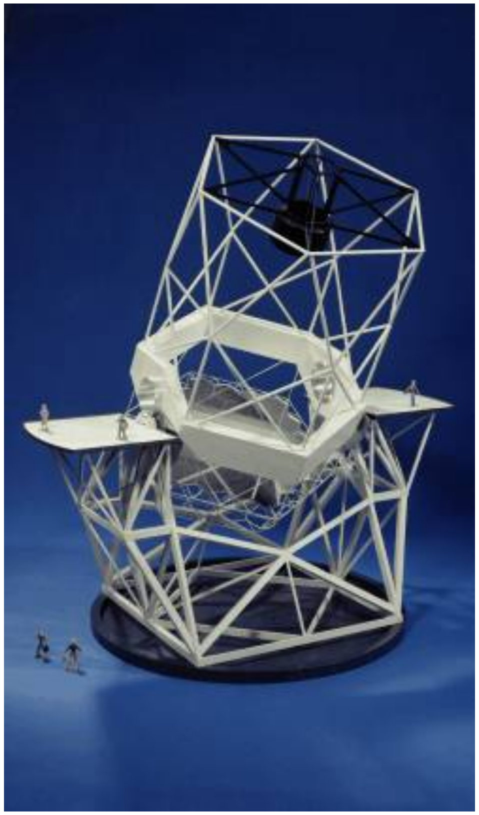


Materials

Properties
Mechanics

Why we need to know about materials

- Stuff is made of stuff
  - what should your part be made of?
  - what does it have to do?
  - how thick should you make it
- The properties we usually care about are:
  - stiffness
  - electrical conductivity
  - thermal conductivity
  - heat capacity
  - coefficient of thermal expansion
  - density
  - hardness, damage potential
  - machine-ability
  - surface condition
  - suitability for coating, plating, etc.

Electrical Resistivity

- Expressed as ρ in $\Omega\cdot$m
  - resistance = $\rho\cdot L/A$
    - where L is length and A is area
  - conductivity is $1/\rho$

Material	ρ ($\times 10^{-6}$ $\Omega\cdot$m)	comments
Silver	0.0147	$$
Gold	0.0219	$$$$
Copper	0.0382	cheapest good conductor
Aluminum	0.047	
Stainless Steel	0.06–0.12	

Thermal Conductivity

- Expressed as κ in W m^{-1} K^{-1}
  - power transmitted = $\kappa \cdot A \cdot \Delta T/t$,
    - where A is area, t is thickness, and ΔT is the temperature across the material

Material	κ (W m^{-1} K^{-1})	comments
Silver	422	room T metals feel cold
Copper	391	great for pulling away heat
Gold	295	
Aluminum	205	
Stainless Steel	10–25	why cookware uses S.S.
Glass, Concrete,Wood	0.5–3	buildings
Many Plastics	~0.4	room T plastics feel warm
G-10 fiberglass	0.29	strongest insulator choice
Stagnant Air	0.024	but usually moving…
Styrofoam	0.01–0.03	can be better than air!

Specific Heat (heat capacity)

- Expressed as c_p in J kg^{-1} K^{-1}
  - energy stored = $c_p \cdot m \cdot \Delta T$
    - where m is mass and ΔT is the temperature change

Material	c_p (J kg^{-1} K^{-1})	comments
water	4184	powerhouse heat capacitor
alcohol (and most liquids)	2500	
wood, air, aluminum, plastic	1000	most things!
brass, copper, steel	400	
platinum	130	

Coefficient of Thermal Expansion

- Expressed as $\alpha = \delta L/L$ per degree K
  - length contraction = $\alpha \cdot \Delta T \cdot L$,
    - where ΔT is the temperature change, and L is length of material

Material	α ($\times 10^{-6}$ K^{-1})	comments
Most Plastics	~100	
Aluminum	24	
Copper	20	
Steel	15	
G-10 Fiberglass	9	
Wood	5	
Normal Glass	3–5	
Invar (Nickel/Iron alloy)	1.5	best structural choice
Fused Silica Glass	0.6	

Density

- Expressed as $\rho = m/V$ in kg·m^{-3}

Material	ρ (kg m^{-3})	comments
Platinum	21452	
Gold	19320	tell this to Indiana Jones
Lead	11349	
Copper, Brass, Steels	7500–9200	
Aluminum Alloys	2700–2900	
Glass	2600	glass and aluminum v. similar
G-10 Fiberglass	1800	
Water	1000	
Air at STP	1.3	

Stress and Strain

- Everything is a spring!
  - nothing is *infinitely* rigid
- You know Hooke's Law:

  $F = k \cdot \delta L$
  - where k is the spring constant (N/m), δL is length change
  - for a given material, k should be proportional to A/L
  - say $k = E \cdot A/L$, where E is some elastic constant of the material
- Now divide by cross-sectional area

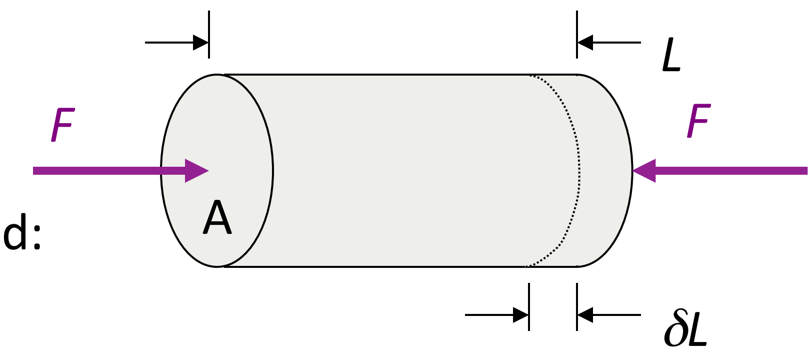
  $F/A = \sigma = k \cdot \delta L/A = E \cdot \varepsilon$ $\qquad \sigma = E \cdot \varepsilon$
  - where ε is $\delta L/L$: the fractional change in length
- This is the stress-strain law for materials
  - σ is the *stress*, and has units of pressure
  - ε is the *strain*, and is unitless

Stress and Strain, Illustrated

- A bar of material, with a force F applied, will change its size by:

  $\delta L/L = \varepsilon = \sigma/E = F/AE$

- Strain is a very useful number, being dimensionless
- Example: Standing on an aluminum rod:
  - $E = 70\times10^{9}$ N·m^{-2} (Pa)
  - say area is 1 cm^2 = 0.0001 m^2
  - say length is 1 m
  - weight is 700 N
  - $\sigma = 7\times10^{6}$ N/m^2
  - $\varepsilon = 10^{-4} \rightarrow \delta L = 100\ \mu$m
  - compression is width of human hair



$$\sigma = F/A$$

$$\varepsilon = \delta L/L$$

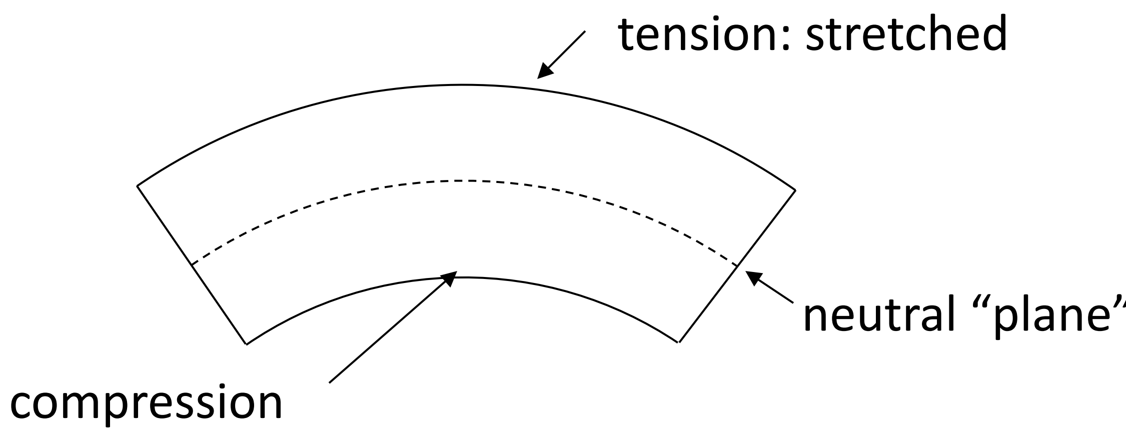
$$\sigma = E\cdot\varepsilon$$

Elastic Modulus

- Basically like a spring constant
  - for a hunk of material, $k = E(A/L)$, but E is the only part of this that is intrinsic to the material: the rest is geometry
- Units are N/m^2, or a pressure (Pascals)

Material	E (GPa)
Tungsten	350
Steel	190–210
Brass, Bronze, Copper	100–120
Aluminum	70
Glass	50–80
G-10 fiberglass	16
Wood	6–15
most plastics	2–3

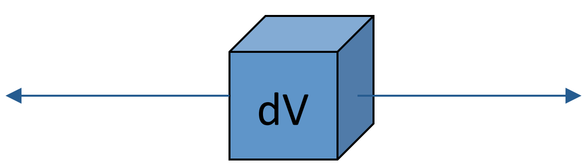
Bending Beams

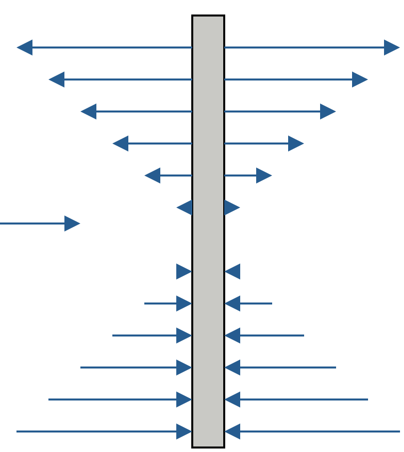


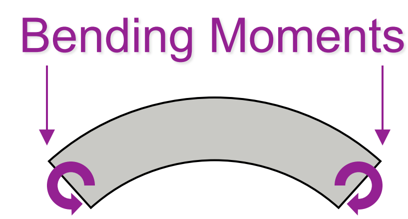
- A bent beam has a stretched outer surface, a compressed inner surface, and a neutral surface somewhere between
- If the neutral length is L, and neutral radius is R, then the strain at some distance, y, from the neutral surface is $(R + y)/R - 1$
  - $\varepsilon = y/R$
  - because arclength for same $\Delta\theta$ is proportional to radius
  - note $L = R\Delta\theta$
- So stress at y is $\sigma = Ey/R$

In the Moment

- Since each mass/volume element is still, the net force is zero
  - Each unit pulls on its neighbor with same force its neighbor pulls on it, and on down the line
  - Thus there is no net moment (torque) on a mass element, and thus on the whole beam
    - otherwise it would rotate: angular momentum would change
  - But something is exerting the bending influence







And we call this "something" the moment (balanced)

What's it take to bend it?

- At each infinitesimal cross section in rod with coordinates (x, y) and area $dA = dxdy$:
  - $dF = \sigma dA = (Ey/R)dA$
  - where y measures the distance from the neutral surface
  - the moment (torque) at the cross section is just $dM = y \cdot dF$
  - so $dM = Ey^2 dA/R$
  - integrating over cross section:

$$M = \int \frac{E}{R} y^2 dxdy = \frac{EI}{R}$$

  - where we have defined the "moment of inertia" as

$$I \equiv \int y^2 dxdy$$

Energy in the bent beam

- We know the force on each volume element:
  - $dF = \sigma \cdot dA = E \cdot \varepsilon \cdot dA = (Ey/R)dA$
- We know that the length changes by $\delta L = \varepsilon dz = \sigma \cdot dz/E$
- So energy is:

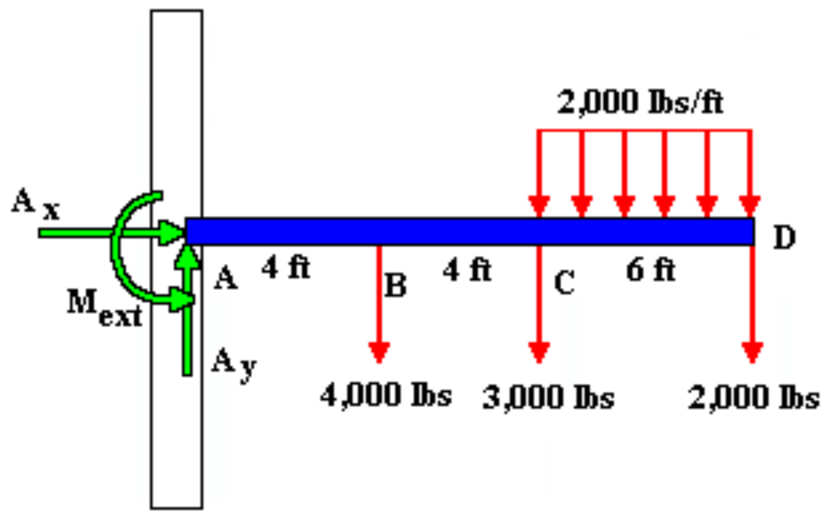
  z-direction
  - $dW = dF \cdot \delta L = dF \cdot \varepsilon \cdot dz = E \cdot \varepsilon \cdot dA \times \varepsilon \cdot dz = E(y/R)^2 dxdydz$
- Integrate this throughout volume

$$W = \frac{E}{R^2} \int y^2 dxdydz = \frac{EIL}{R^2}$$

- So $W = M(L/R) \approx M\theta \propto \theta^2$
  - where θ is the angle through which the beam is bent

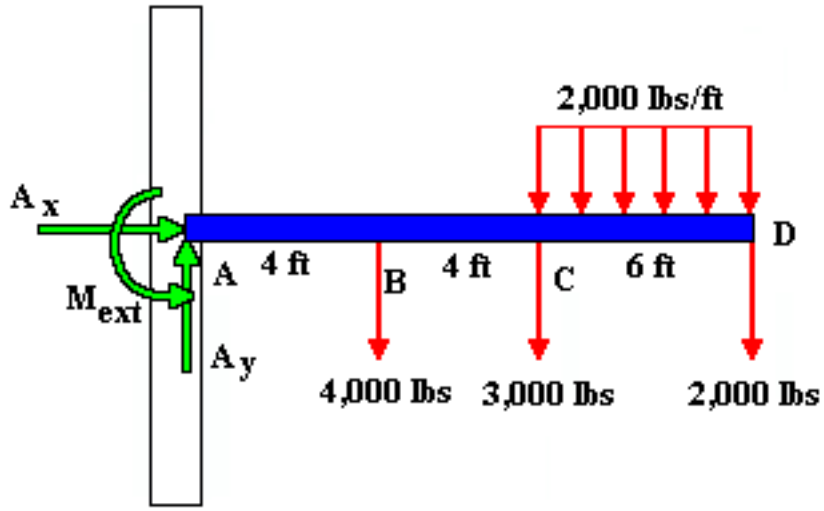
Calculating beam deflection

- We start by making a free-body diagram so that all forces and torques are balanced
  - otherwise the beam would fly/rotate off in some direction



  - In this case, the wall exerts forces and moments on the beam (though A_x=0)
  - This example has three point masses and one distributed load

Tallying the forces/moments

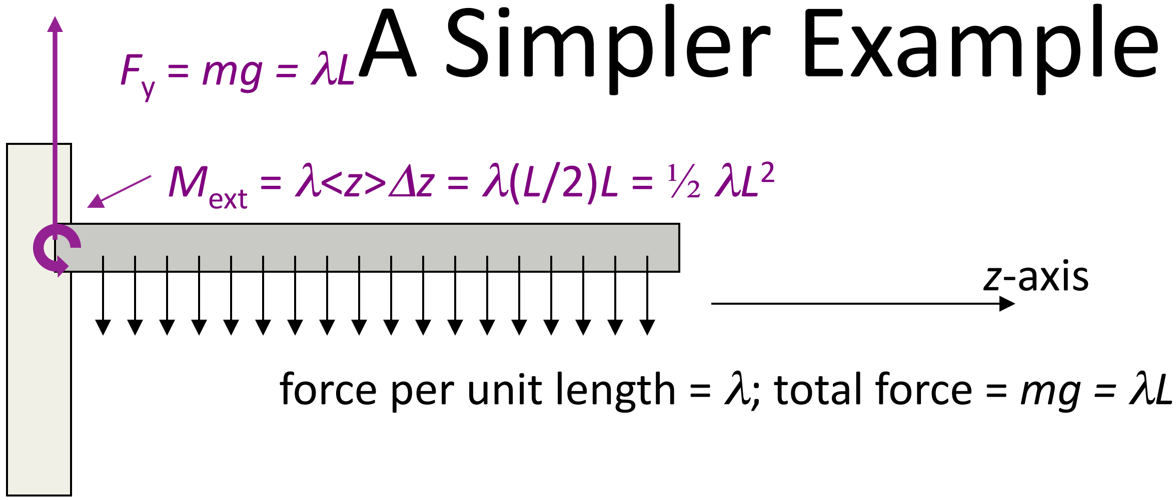


- $A_x = 0$; $A_y = 21{,}000$ lbs
- $M_{ext} = (4)(4000) + (8)(3000) + (14)(2000) + (11)(6)(2000) = 200{,}000$ ft-lbs
  - last term is integral:

$$M = \int_{x_1}^{x_2} \lambda x dx = \left[\lambda \frac{x^2}{2}\right]_{x_1}^{x_2} = \lambda \frac{x_1 + x_2}{2}(x_2 - x_1) = \lambda \langle x \rangle \Delta x$$

  - where λ is the force per unit length (2000 lbs/ft)

A Simpler Example

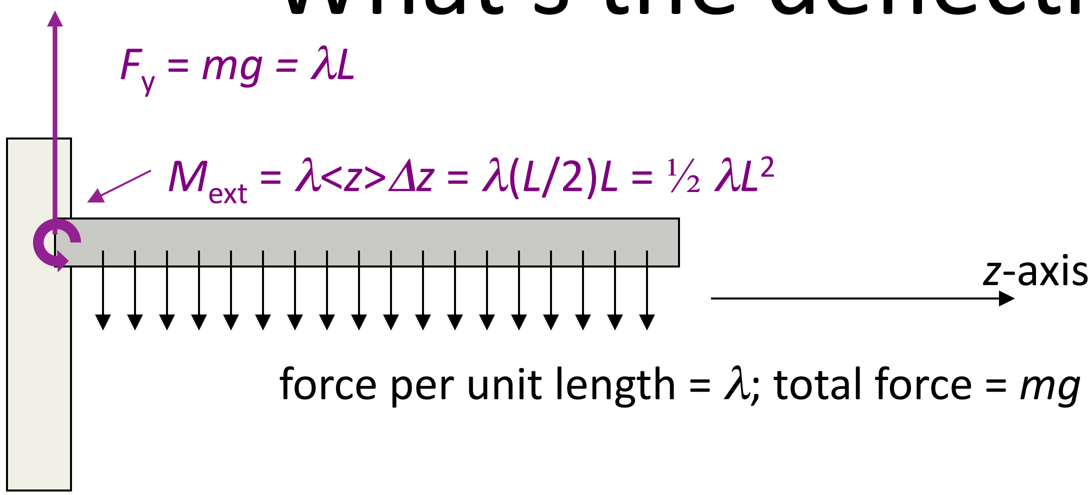


- A cantilever beam under its own weight (or a uniform weight)
  - F_y and M_{ext} have been defined above to establish force/moment balance
  - At any point, distance z along the beam, we can sum the moments about this point and find:

$$M_{\mathrm{tot}} = M_{\mathrm{ext}} - zF_y + \int_0^L \lambda(z - z')dz' = \frac{1}{2}\lambda L^2 - \lambda L z + \lambda L z - \frac{1}{2}\lambda L^2 = 0$$

  - validating that we have no net moment about any point, and thus the beam will not spin up on its own!

What's the deflection?



- At any point, z, along the beam, the unsupported moment is given by:

$$M(z) = \int_z^L \lambda(z - z')dz' = \lambda\left[Lz - z^2 - \frac{L^2}{2} + \frac{z^2}{2}\right] = -\frac{mg}{2L}(z^2 - 2Lz + L^2)$$

- From before, we saw that moment and radius of curvature for the beam are related:
  - $M = EI/R$
- And the radius of a curve, Y, is the reciprocal of the second derivative:
  - $d^2Y/dz^2 = 1/R = M/EI$
  - so for this beam, $d^2Y/dz^2 = M/EI =$ $-\frac{mg}{2EIL}(z^2 - 2Lz + L^2)$

Calculating the curve

- If we want to know the deflection, *Y*, as a function of distance, *z*, along the beam, and have the second derivative…
- Integrate the second derivative twice:

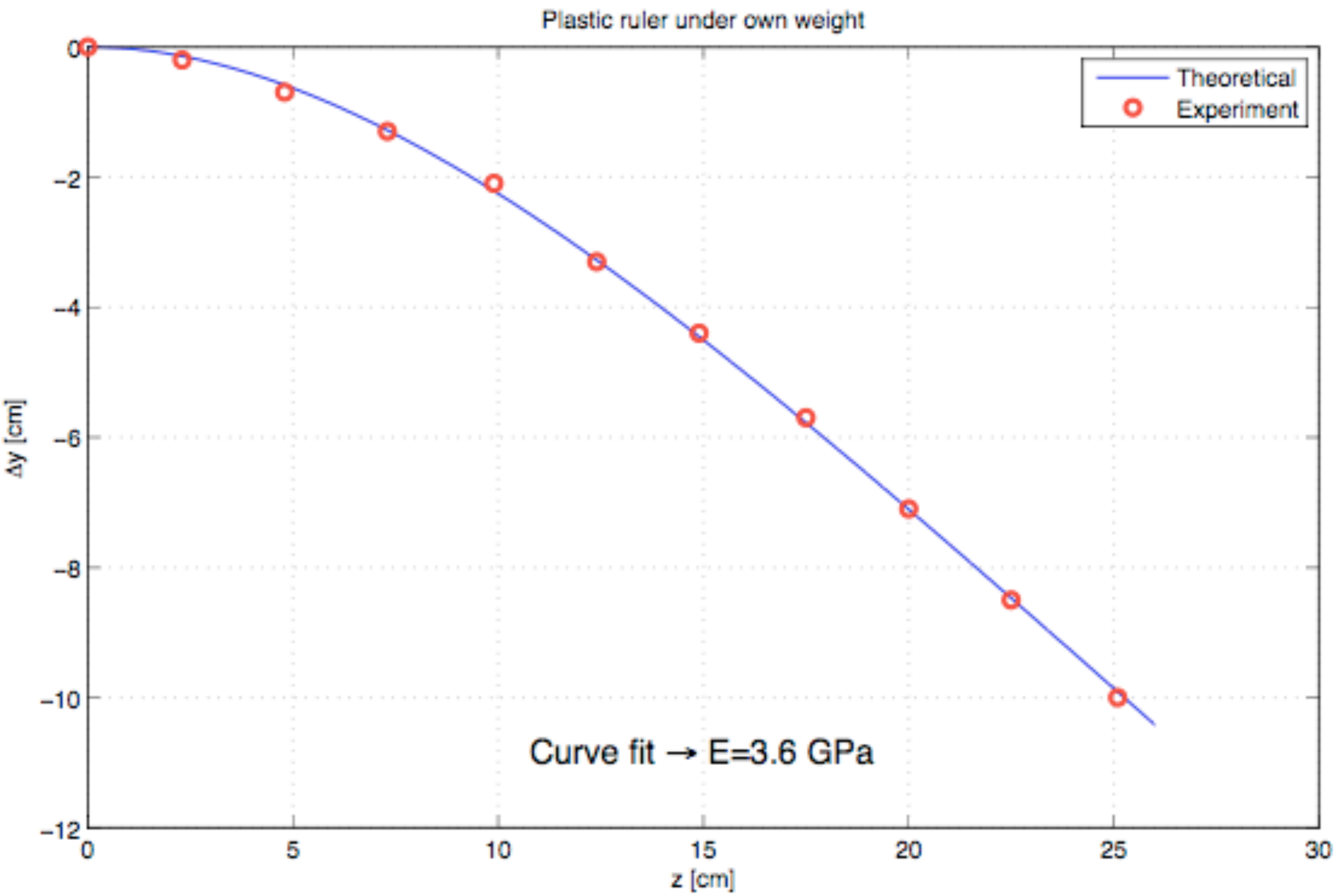
$$\frac{d^2Y}{dz^2} = -\frac{mg}{2EIL}(z^2 - 2Lz + L^2) \rightarrow Y = -\frac{mg}{2EIL}\left(\frac{z^4}{12} - \frac{Lz^3}{3} + \frac{L^2z^2}{2} + Cz + D\right)$$

  - where *C* and *D* are constants of integration
  - at *z*=0, we define *Y*=0, and note the slope is zero, so *C* and *D* are likewise zero
  - so, the beam follows:

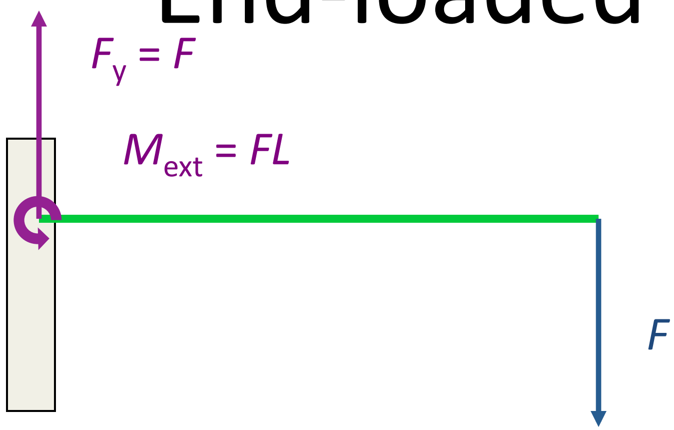
$$Y = -\frac{mg}{24EIL}\left(z^4 - 4Lz^3 + 6L^2z^2\right)$$

  - with maximum deflection at end: $Y_{\max} = \frac{mgL^3}{8EI}$

Bending Curve, Illustrated

- 

End-loaded cantilever beam



- Playing the same game as before (integrate moment from z to L):

$$M(z) = (z-L)F \rightarrow \frac{d^2Y}{dz^2} = \frac{1}{R(z)} = \frac{M(z)}{EI} = \frac{F}{EI}(z-L)$$

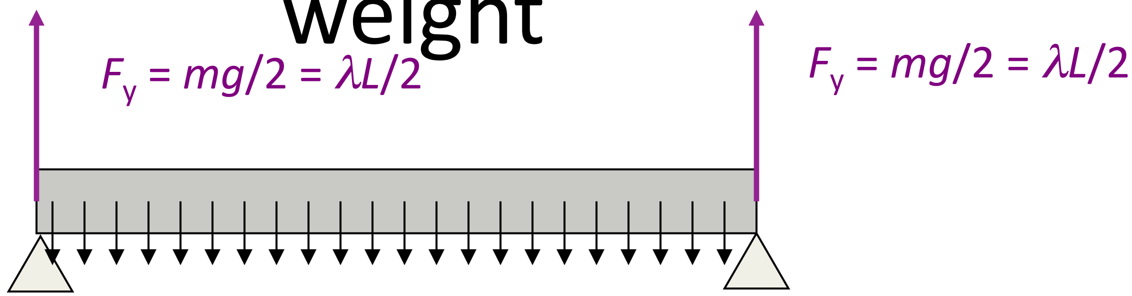
  – which integrates to:

$$Y = \frac{F}{EI}\left(\frac{z^3}{6} - \frac{Lz^2}{2} + Cz + D\right)$$

  – and at z=0, Y=0 and slope=0 → $C = D = 0$, yielding:

$$Y = \frac{F}{6EI}(z^3 - 3Lz^2) \qquad Y_{\max} = \frac{FL^3}{3EI}$$

Simply-supported beam under own weight



force per unit length = λ; total force = $mg = \lambda L$

- This support cannot exert a moment

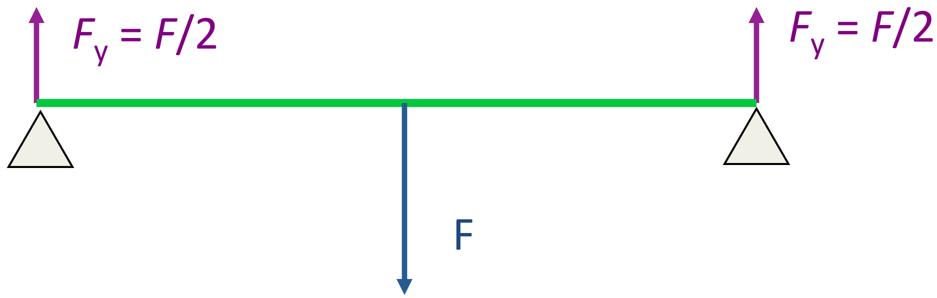
$$M(z) = \int_z^L \lambda(z - z')dz' + \frac{1}{2}\lambda L(L - z) = \frac{1}{2}\lambda(Lz - z^2)$$

$$\frac{d^2Y}{dz^2} = \frac{\lambda}{2EI}(Lz - z^2) \rightarrow Y = \frac{\lambda}{2EI}\left(\frac{Lz^3}{6} - \frac{z^4}{12} + Cz + D\right)$$

  – at z=0, Y=0 → D = 0; at z=$L/2$, slope = 0 → $C = -L^3/12$

$$Y = \frac{mg}{24EIL}\left(2Lz^3 - z^4 - L^3 z\right) \qquad Y_{\max} = \frac{5}{384}\frac{mgL^3}{EI}$$

Simply-supported beam with centered weight



- Working only from $0 < z < L/2$ (symmetric):

$$M(z) = F\left(z - \frac{L}{2}\right) + \frac{F}{2}(L - z) = \frac{Fz}{2} \rightarrow \frac{d^2Y}{dz^2} = \frac{Fz}{2EI}$$

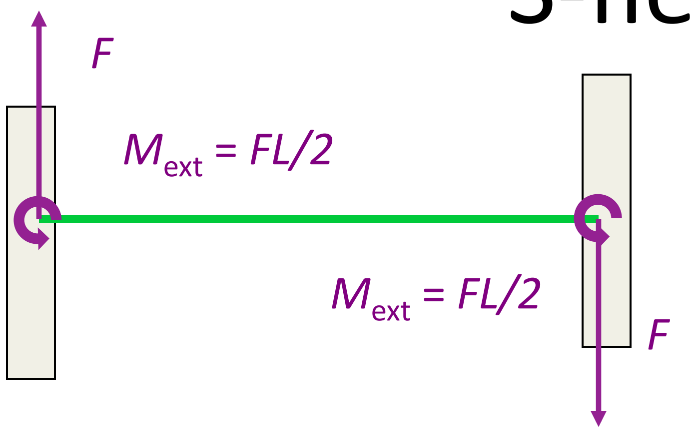
  – integrating twice, setting $Y(0) = 0$, $Y'(L/2) = 0$:

$$Y = \frac{F}{12EI}(z^3 + Cz + D) \rightarrow Y = \frac{F}{12EI}\left(z^3 - \frac{3L^2 z}{4}\right)$$

  – and the max deflection (at $z=L/2$):

$$Y_{\max} = \frac{FL^3}{48EI}$$

S-flex beam



"walls" are held vertical; beam flexes in "S" shape

total M(z) = 2M_{ext} - Fz - $F(L-z)$ = 0 for all z

- Playing the same game as before (integrate moment from z to L):

$$M(z) = M_{\rm ext} - F(L-z) = Fz - \frac{FL}{2} \rightarrow \frac{d^2Y}{dz^2} = \frac{1}{R(z)} = \frac{M(z)}{EI} = \frac{F}{2EI}(2z - L)$$

  – which integrates to:

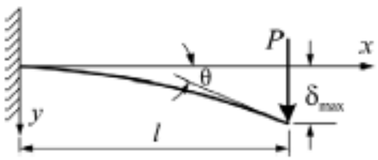
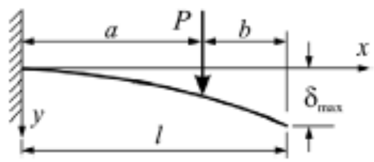
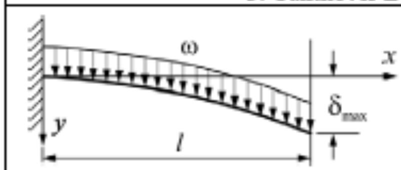
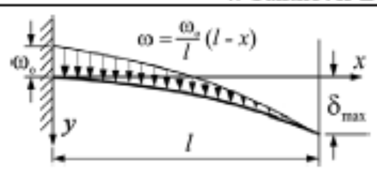
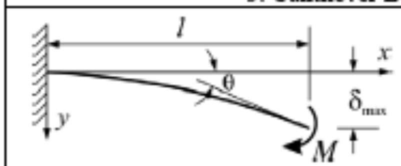
$$Y = \frac{F}{EI}\left(\frac{z^3}{6} - \frac{Lz^2}{4} + Cz + D\right)$$

  – and at z=0, Y=0 and slope=0 → C = D = 0, yielding:

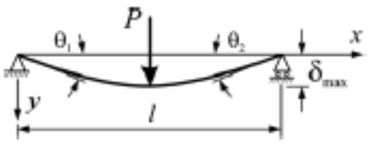
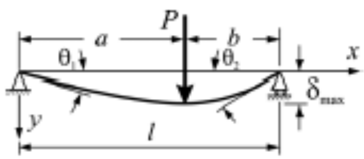
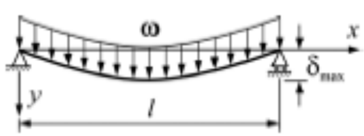
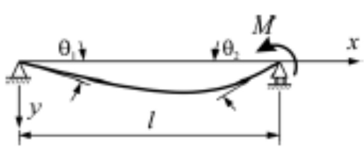
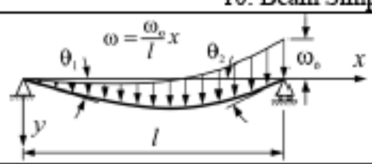
$$Y = \frac{F}{2EI}\left(\frac{z^3}{3} - \frac{Lz^2}{2}\right) \qquad Y'(L) = 0 \qquad Y_{\rm max} = \frac{FL^3}{12EI}$$

as it should be

Cantilevered beam formulae

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \dfrac{Pl^2}{2EI}$	$y = \dfrac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \dfrac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load P at any point			
	$\theta = \dfrac{Pa^2}{2EI}$	$y = \dfrac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \dfrac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \dfrac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \dfrac{\omega l^3}{6EI}$	$y = \dfrac{\omega x^2}{24EI}\left(x^2 + 6l^2 - 4lx\right)$	$\delta_{max} = \dfrac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \dfrac{\omega_o l^3}{24EI}$	$y = \dfrac{\omega_o x^2}{120 lEI}\left(10l^3 - 10l^2x + 5lx^2 - x^3\right)$	$\delta_{max} = \dfrac{\omega_o l^4}{30EI}$
5. Cantilever Beam – Couple moment M at the free end			
	$\theta = \dfrac{Ml}{EI}$	$y = \dfrac{Mx^2}{2EI}$	$\delta_{max} = \dfrac{Ml^2}{2EI}$

Simply Supported beam formulae

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \dfrac{Pl^2}{16EI}$	$y = \dfrac{Px}{12EI}\left(\dfrac{3l^2}{4} - x^2\right)$ for $0 < x < \dfrac{l}{2}$	$\delta_{max} = \dfrac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \dfrac{Pb(l^2-b^2)}{6lEI}$ $\theta_2 = \dfrac{Pab(2l-b)}{6lEI}$	$y = \dfrac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \dfrac{Pb}{6lEI}\left[\dfrac{l}{b}(x-a)^3 + \left(l^2-b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \dfrac{Pb\left(l^2-b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{\left(l^2-b^2\right)/3}$ $\delta = \dfrac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \dfrac{\omega l^3}{24EI}$	$y = \dfrac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \dfrac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment M at the right end			
	$\theta_1 = \dfrac{Ml}{6EI}$ $\theta_2 = \dfrac{Ml}{3EI}$	$y = \dfrac{Mlx}{6EI}\left(1 - \dfrac{x^2}{l^2}\right)$	$\delta_{max} = \dfrac{Ml^2}{9\sqrt{3}\,EI}$ at $x = \dfrac{l}{\sqrt{3}}$ $\delta = \dfrac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \dfrac{7\omega_o l^3}{360EI}$ $\theta_2 = \dfrac{\omega_o l^3}{45EI}$	$y = \dfrac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\dfrac{\omega_o l^4}{EI}$ at $x = 0.519\,l$ $\delta = 0.00651\dfrac{\omega_o l^4}{EI}$ at the center

Lessons to be learned

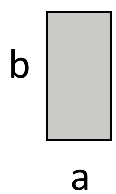
- All deflections inversely proportional to E
  - the stiffer the spring, the less it bends
- All deflections inversely proportional to I
  - cross-sectional geometry counts
- All deflections proportional to applied force/weight
  - in linear regime: Hooke's law
- All deflections proportional to length cubed
  - pay the price for going long!
  - beware that if beam under own weight, $mg \propto L$ also (so L^4)
- Numerical prefactors of maximum deflection, Y_{max}, for same load/length were:
  - 1/3 for end-loaded cantilever
  - 1/8 for uniformly loaded cantilever
  - 1/48 for center-loaded simple beam
  - 5/384 ~ 1/77 for uniformly loaded simple beam
- Thus support at both ends helps: cantilevers suffer

Getting a feel for the *I*-thingy

- The "moment of inertia," or second moment came into play in every calculation

$$I \equiv \int y^2 dx dy$$

- Calculating this for a variety of simple cross sections:
- Rectangular beam:

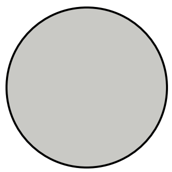


$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dy = a \left[\frac{y^3}{3} \right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{ab^3}{12} = \frac{A^2}{12} \frac{b}{a}$$

  – note the cube-power on *b*: twice as thick (in the direction of bending) is 8-times better!
  – For fixed area, win by fraction *b/a*

Moments Later

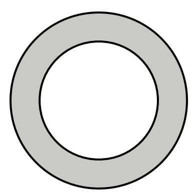
- Circular beam
  - work in polar coordinates, with $y = r\sin\theta$



radius, R

$$I = \int_0^R r dr \int_0^{2\pi} r^2 \sin^2\theta d\theta = \frac{\pi R^4}{4} = \frac{A^2}{4\pi}$$

  - note that the area-squared fraction ($1/4\pi$) is very close to that for a square beam (1/12 when $a = b$)
  - so for the same area, a circular cross section performs almost as well as a square
- Circular tube

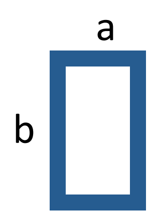


inner radius R_1, outer radius R_2
or, outer radius R, thickness t

$$I = \int_{R_1}^{R_2} r dr \int_0^{2\pi} r^2 \sin^2\theta d\theta = \frac{\pi}{4}(R_2^4 - R_1^4) = \frac{\pi}{4}(R_2^2 + R_1^2)(R_2^2 - R_1^2) = \frac{A}{4}(R_1^2 + R_2^2)$$

And more moments

- Circular tube, continued
  - if $R_2 = R$, $R_1 = R\text{-}t$, for small t: $I \approx (A^2/4\pi)(R/t)$
  - for same area, thinner wall stronger (until crumples/dents compromised integrity)
- Rectangular Tube
  - wall thickness = t



$$I = 2\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{\frac{b}{2}-t}^{\frac{b}{2}} y^2 dy + 2\int_{\frac{a}{2}-t}^{\frac{a}{2}} dx \int_{-\frac{b}{2}+t}^{\frac{b}{2}-t} y^2 dy = 2a\left[\frac{b^3}{24} - \frac{(\frac{b}{2}-t)^3}{3}\right] + 4t\frac{(\frac{b}{2}-t)^3}{3}$$

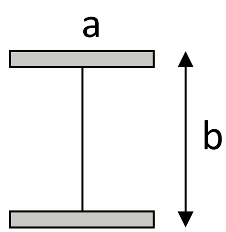
  - and if t is small compared to *a* & *b*:

$$I \approx \frac{ab^2t}{2} + \frac{b^3t}{6}$$ and for a square geom.: $$I_{\text{sq}} \approx \frac{2a^3t}{3} \approx \frac{A^2}{24}\frac{a}{t}$$

  - note that for $a = b$ (square), side walls only contribute 1/4 of the total moment of inertia: best to have more mass at larger *y*-value: this is what makes the integral bigger!

The final moment

- The I-beam
  - we will ignore the minor contribution from the “web” connecting the two flanges



$$I = 2\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{\frac{b}{2}-t}^{\frac{b}{2}} y^2 dy = 2a\left[\frac{b^3}{24} - \frac{(\frac{b}{2}-t)^3}{3}\right] \approx \frac{ab^2t}{2}$$

  - note this is just the rectangular tube result without the side wall. If you want to put a web member in, it will add an extra $b^3t/12$, roughly
  - in terms of area = $2at$: $I \approx \frac{A^2}{8}\frac{b}{a}\frac{b}{t}$
- The I-beam puts as much material at high y-value as it can, where it maximally contributes to the beam stiffness
  - the web just serves to hold these flanges apart

Lessons on moments

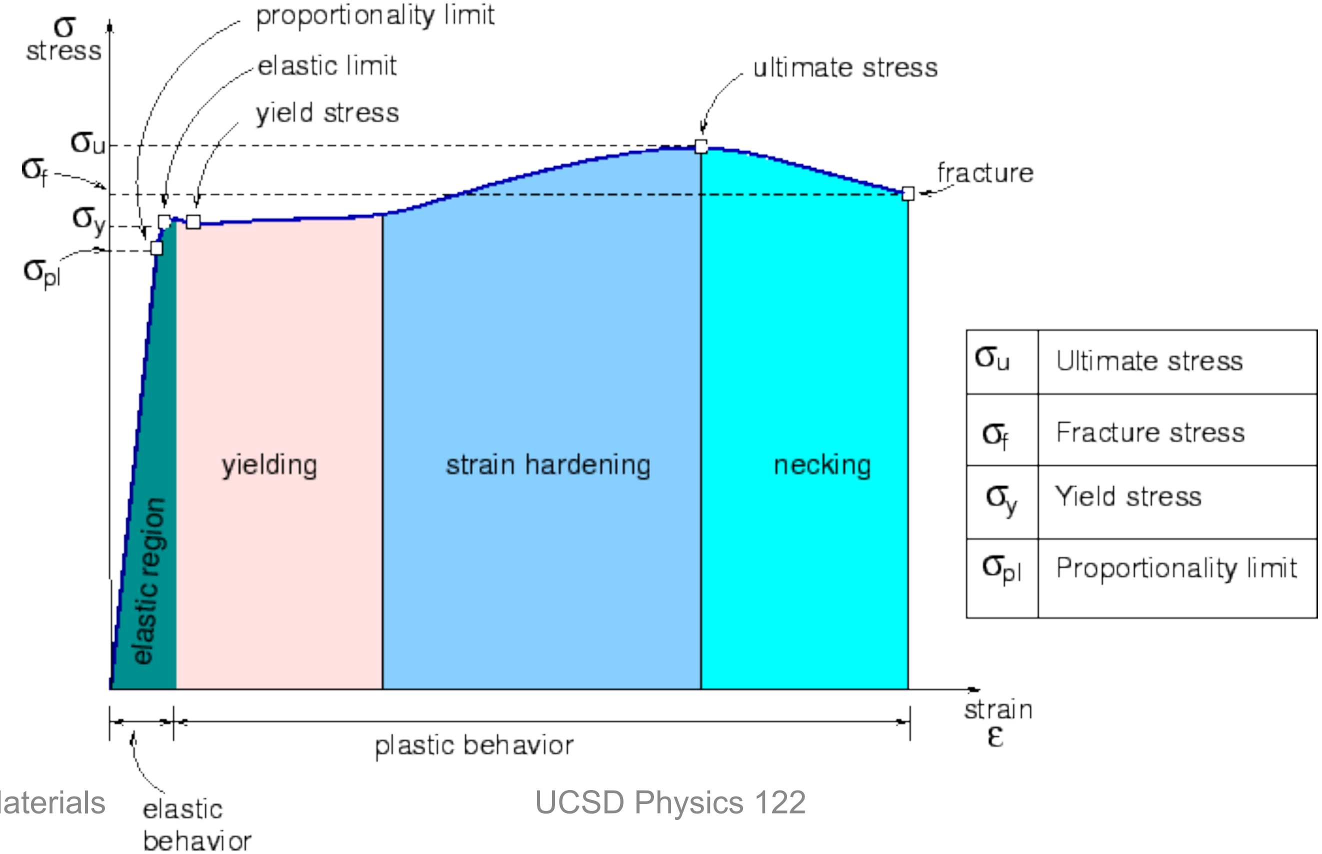
- Thickness in the direction of bending helps to the third power
  - always orient a 2×4 with the "4" side in the bending direction
- For their weight/area, tubes do better by putting material at high *y*-values
- I-beams maximize the moment for the same reason
- For square geometries, equal material area, and a thickness 1/20 of width (where appropriate), we get:
  - square solid: $I \approx A^2/12 \approx 0.083A^2$
  - circular solid: $I \approx A^2/4\pi \approx 0.080A^2$
  - square tube: $I \approx 20A^2/24 \approx 0.83A^2$
  - circular tube: $I \approx 10A^2/4\pi \approx 0.80A^2$
  - I-beam: $I \approx 20A^2/8 \approx 2.5A^2$
- I-beam wins hands-down

10× better than solid form

func. of assumed 1/20 ratio

Beyond Elasticity

- Materials remain elastic for a while
  - returning to exact previous shape
- But ultimately plastic (permanent) deformation sets in
  - and without a great deal of extra effort



Breaking Stuff

- Once out of the elastic region, permanent damage results
  - thus one wants to stay below the yield stress
  - yield strain = yield stress / elastic modulus

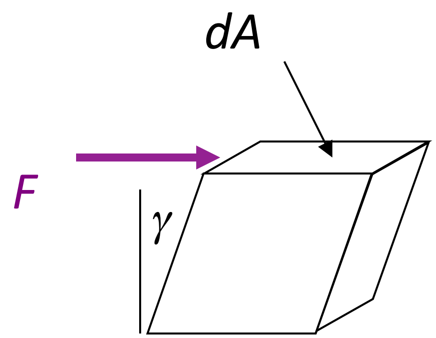
Material	Yield Stress (MPa)	Yield Strain
Tungsten*	1400	0.004
Steel	280–1600	0.0015–0.0075
Brass, Bronze, Copper	60–500	0.0005–0.0045
Aluminum	270–500	0.004–0.007
Glass*	70	0.001
Wood	30–60	0.0025–0.005
most plastics*	40–80	0.01–0.04

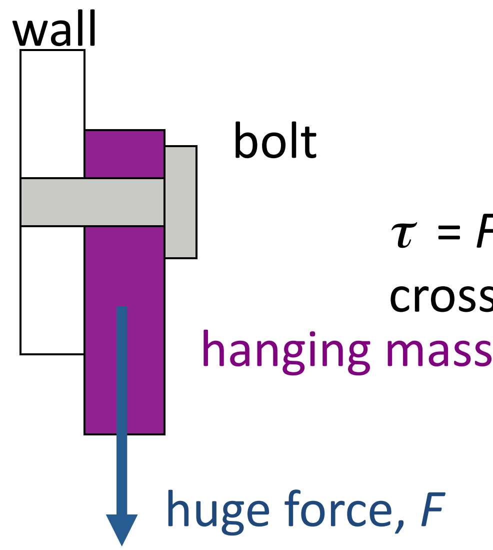
* ultimate stress quoted (see next slide for reason)

Notes on Yield Stress

- The entries in red in the previous table represent ultimate stress rather than yield stress
  – these are materials that are brittle, experiencing no plastic deformation, or plastics, which do not have a well-defined elastic-to-plastic transition
- There is much variability depending on alloys
  – the yield stress for steels are
    - stainless: 280–700
    - machine: 340–700
    - high strength: 340–1000
    - tool: 520
    - spring: 400–1600 (want these to be elastic as long as possible)
  – aluminum alloys
    - 6061-T6: 270 (most commonly used in machine shops)
    - 7075-T6: 480

Shear Stress





τ = *F/A*, where *A* is bolt's cross-sectional area

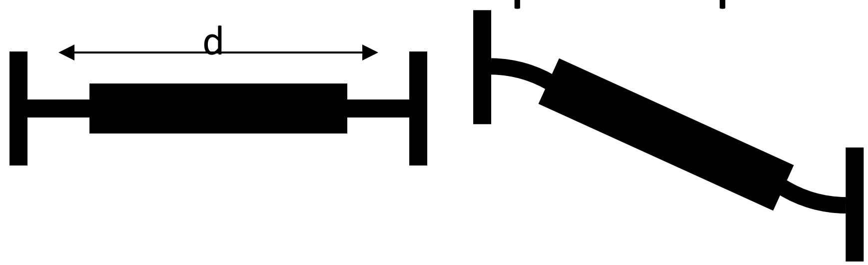
- $\tau = G\gamma$
  - τ is the shear stress ($N{\cdot}m^{-2}$) = force over area = *F/dA*
    - *dA* is now the shear plane (see diagram)
  - *G* is the shear modulus ($N{\cdot}m^{-2}$)
  - γ is the angular deflection (radians)
- The shear modulus is related to *E*, the elastic modulus
  - $E/G = 2(1+\nu)$
  - ν is called Poisson's ratio, and is typically around 0.27–0.33

Practical applications of stress/strain

- Infrared spectrograph bending (flexure)
  - dewar whose inner shield is an aluminum tube 1/8 inch (3.2 mm) thick, 5 inch (127 mm) radius, and 1.5 m long
  - weight is 100 Newtons
  - loaded with optics throughout, so assume (extra) weight is 20 kg → 200 Newtons
  - If gravity loads sideways (when telescope is near horizon), what is maximum deflection, and what is maximum angle?
  - calculate $I \approx (A^2/4\pi)(R/t) = 2\times10^{-5}$ m^4
  - $E = 70\times10^9$
  - $Y_{max} = mgL^3/8EI = 90$ μm deflection
  - $Y'_{max} = mgL^2/6EI = 80$ μR angle
- Now the effect of these can be assessed in connection with the optical performance

Applications, continued

- A stainless steel flexure to permit parallel displacement



  - each flexing member has length L = 13 mm, width a = 25 mm, and bending thickness b = 2.5 mm, separated by d = 150 mm
  - how much range of motion do we have?
  - stress greatest on skin (max tension/compression)
  - Max strain is $\varepsilon = \sigma_y/E$ = 280 MPa / 200 GPa = 0.0014
  - strain is y/R, so $b/2R = 0.0014 \rightarrow R = b/0.0028 = 0.9$ m
  - $\theta = L/R = 0.013/0.9 = 0.014$ radians (about a degree)
  - so max displacement is about $d\cdot\theta$ = 2.1 mm
  - energy in bent member is EIL/R^2 = 0.1 J per member $\rightarrow$ 0.2 J total
  - $W = F\cdot d \rightarrow F$ = (0.2 J)/(0.002 m) = 100 N (~ 20 lb)

Flexure Design

- Sometimes you need a design capable of flexing a certain amount without breaking, but want the thing to be as stiff as possible under this deflection
  - strategy:
    - work out deflection formula;
    - decide where maximum stress is (where moment, and therefore curvature, is greatest);
    - work out formula for maximum stress;
    - combine to get stress as function of displacement
    - invert to get geometry of beam as function of tolerable stress
  - example: end-loaded cantilever

$$Y_{\max} = \frac{FL^3}{3EI}$$

Δy is displacement from centerline (half-thickness)

$$M(z) = F(z - L) \rightarrow \max \text{ at } z = 0$$

$$\text{max strain, } \varepsilon = \frac{\Delta y}{R} = \frac{\Delta y M_{\max}}{EI} = \frac{FL\Delta y}{EI} \rightarrow \text{max stress, } \sigma_{\max} = E\varepsilon = \frac{FL\Delta y}{I}$$

Flexure Design, cont.

- Note that the ratio *F/I* appears in both the Y_{max} and σ_{max} formulae (can therefore eliminate)

$$\sigma_{\mathrm{max}} = \frac{F}{I} L\Delta y = \frac{3EY_{\mathrm{max}}}{L^3} L\Delta y = \frac{3EY_{\mathrm{max}}\Delta y}{L^2} = \frac{3EY_{\mathrm{max}}h}{2L^2}$$

where $h = 2\Delta y$ is beam thickness

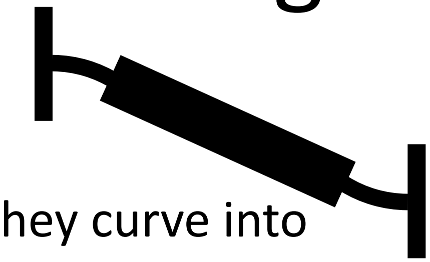
- If I can tolerate some fraction of the yield stress

$\sigma_{max} = \sigma_y/\Phi$, where Φ is the safety factor (often chosen to be 2)

$$h = \frac{\sigma_{\mathrm{max}}}{E}\frac{2L^2}{3Y_{\mathrm{max}}} = \frac{\sigma_{\mathrm{y}}}{\Phi E}\frac{2L^2}{3Y_{\mathrm{max}}} = \varepsilon_{\mathrm{max}}\frac{2L^2}{3Y_{\mathrm{max}}}$$

- so now we have the necessary (maximum) beam thickness that can tolerate a displacement Y_{max} without exceeding the safety factor, Φ
- You will need to go through a similar procedure to work out the thickness of a flexure that follows the S-bend type (prevalent in the Lab 2)

Notes on Bent Member Flexure Design



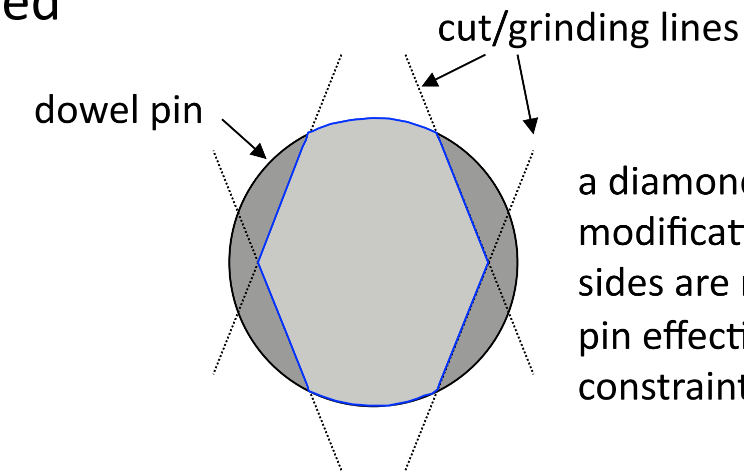
- When the flex members have moments at both ends, they curve into more-or-less an arc of constant radius, accomplishing angle θ
- $R = EI/M$, and $\theta = L/R = ML/EI$, where L is the length of the flexing beam (not the whole assembly)
- $\sigma_{\text{max}} = E\varepsilon_{\text{max}} = E\Delta y/R = h\theta E/2L$, so $h = (\sigma_y/\Phi E)\times(2L/\theta)$
  – where $h = 2\Delta y$ and $R = L/\theta$

Kinematic Design

- Physicists care where things are
  - position and orientation of optics, detectors, etc. can really matter
- Much of the effort in the machine shop boils down to holding things where they need to be
  - and often allowing controlled adjustment around the nominal position
- Any rigid object has 6 degrees of freedom
  - three translational motions in 3-D space
  - three "Euler" angles of rotation
    - take the earth: need to know two coordinates in sky to which polar axis points, plus one rotation angle (time dependent) around this axis to nail its orientation
- Kinematic design seeks to provide minimal/critical constraint

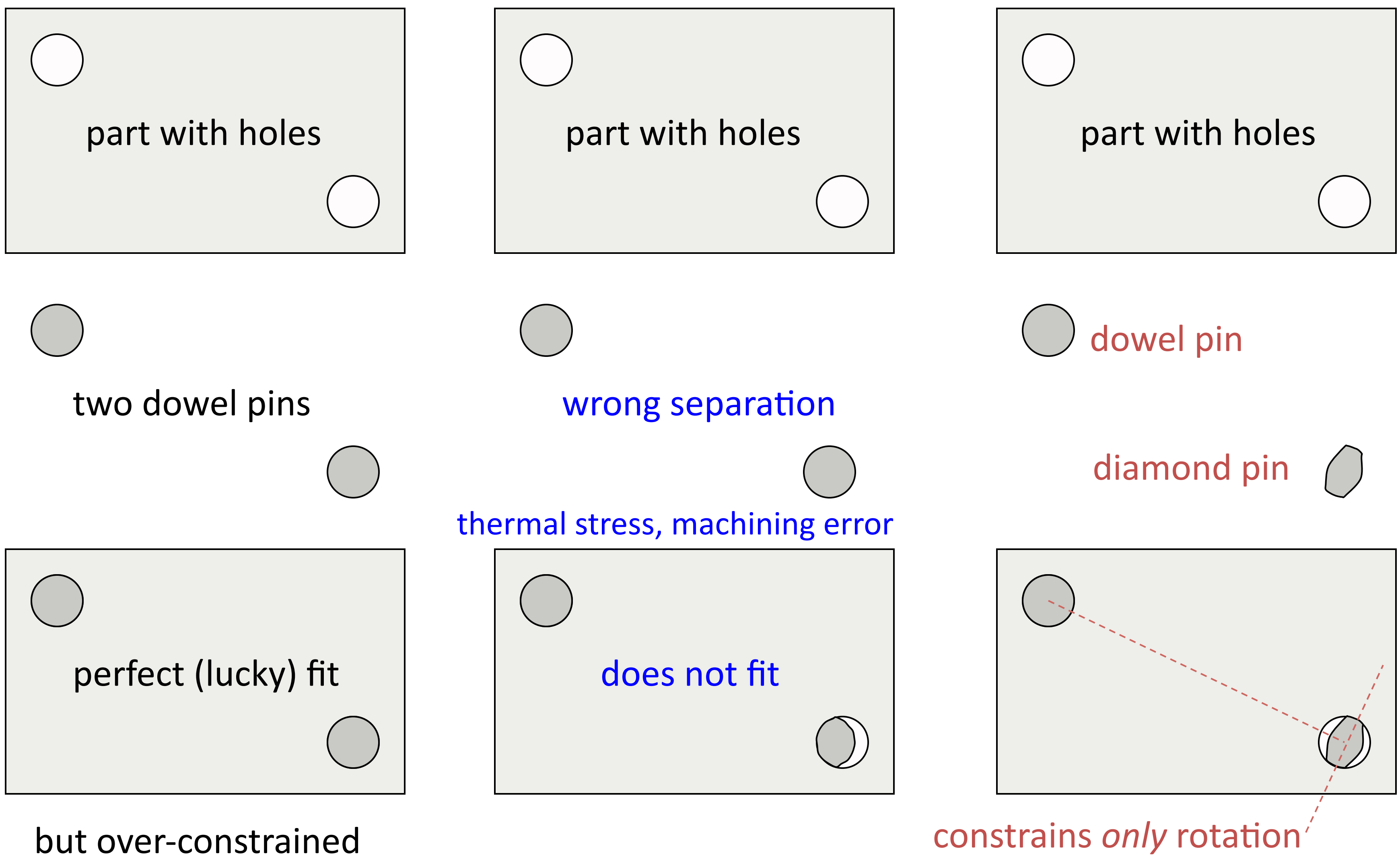
Basic Principles

- A three-legged stool will never rock
  - as opposed to 4-legged
  - each leg removes one degree of freedom, leaving 3
    - can move in two dimensions on planar floor, and can rotate about vertical axis
- A pin & hole constrain two translational degrees of freedom
- A second pin constrains rotation
  - though best if it's a diamond-shaped-pin, so that the device is not over-constrained



a diamond pin is a home-made modification to a dowel pin: sides are removed so that the pin effectively is a one-dim. constraint rather than 2-d

Diamond Pin Idea



diamond pin must be ground on grinder from dowel pin: cannot buy

Kinematic Summary

- Combining these techniques, a part that must be located precisely will:
  - sit on three legs or pads
  - be constrained within the plane by a dowel pin and a diamond pin
- Reflective optics will often sit on three pads
  - when making the baseplate, can leave three bumps in appropriate places
    - only have to be 0.010 high or so
  - use delrin-tipped (plastic) spring plungers to gently push mirror against pads

References and Assignment

- For more on mechanics:
  - *Mechanics of Materials*, by Gere and Timoshenko
- For a boatload of stress/strain/deflection examples worked out:
  - *Roark's Formulas for Stress and Strain*
- Suggested reading from reference text:
  - Section 1.5; 1.5.1 & 1.5.5; 1.6, 1.6.1, 1.6.5, 1.6.6 (3rd ed.)
  - Section 1.2.3; 1.6.1; 1.7 (1.7.1, 1.7.5, 1.7.6) (4th ed.)
- Additional reading on Phys239 website
  - https://tmurphy.physics.ucsd.edu/phys239/lectures/phys239_2016_lec12.pdf
  - very similar development to this lecture, with more text