The Art of Estimation Physics

also known as **Ph 2239**

I. "Natural Units"

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Richard Feynman :

"use sloppy thinking"

"never attempt a physics problem until you know the answer"

"Natural Units"

In this system of units there is only one fundamental dimension, *energy*. This is accomplished by setting Planck's constant, the speed of light, and Boltzmann's constant to unity, *i.e.*,

$$\hbar = c = k_{\rm B} = 1$$

By doing this most any quantity can be expressed as powers of energy, because now we easily can arrange for

$$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}$$

To restore "normal" units we need only insert appropriate powers of of the fundamental constants above

It helps to remember the dimensions of these quantities . . .

$$[\hbar c] = [\text{Energy}] \cdot [\text{Length}]$$

$$[c] = [\text{Length}] \cdot [\text{Time}]^{-1}$$

for example, picking convenient units (*for m*

$$\hbar\, c \approx 197.33 \ \text{MeV fm}$$

$$c \approx 2.9979 \times 10^{23} \ \text{fm}\,\text{s}^{-1}$$

length units

$$1\,\text{fm} = 10^{-13}\,\text{cm} = 10^{-15}\,\text{m}$$

$$1\,\text{Å} = 10^{-8}\,\text{cm} = 10^{-10}\,\text{m} = 0.1\,\text{nm}$$

Figure of merit for typical visible light wavelength $\lambda = 10^4\,\text{Å} = 10^3\,\text{nm}$ and corresponding energy $E = h\nu = 2\pi\hbar\nu = 2\pi\frac{\hbar c}{\lambda}$

$$E = 2\pi\frac{\hbar c}{\lambda} = 2\pi\frac{1.9733\times 10^3\,\text{eV Å}}{10^4\,\text{Å}} \approx 1.24\,\text{eV}$$

Boltzmann's constant

– from now on measure temperature in energy units

$$[k_\text{B}] = \frac{[\text{Energy}]}{\text{Kelvin}}$$

for example . . .

$$k_\text{B} = \frac{8.617\times 10^{-5}\,\text{eV}}{\text{Kelvin}} \sim 10^{-4}\,\frac{\text{eV}}{\text{K}}$$

but I like $k_\text{B} = 0.08617 \quad \text{MeV}/T9$

with $T9 \equiv \frac{T}{10^9\,\text{K}}$

Examples:

Number Density

$$n = \frac{\#}{\text{volume}} = \frac{\#}{[\text{Length}]^3} = [\text{Energy}]^3$$

$$[n] = \text{MeV}^3 = \frac{\text{MeV}^3}{(\hbar c)^3} = \frac{1}{(\text{fm})^3}$$

e.g., number density of photons in thermal equilibrium at temperature T= 1 Me

$$n_\gamma = \frac{2\,\zeta(3)}{\pi^2}T^3 \approx \frac{2\cdot(1.20206)}{\pi^2}T^3 \approx 0.2436\,T^3$$

$$= 0.2436\ \text{MeV}^3 = \frac{0.2436\,\text{MeV}^3}{(\hbar c)^3} = \frac{0.2436\,\text{MeV}^3}{(197.33\,\text{MeV fm})^3} = 3.170\times 10^{-8}\,\text{fm}^{-3}$$

$$\approx 3.17\times 10^{31}\,\text{cm}^{-3}$$

stresses

e.g., energy density, pressure, shear stress, *etc.*

$$[\text{Pressure}] = \frac{[\text{Force}]}{[\text{Area}]} = \frac{[\text{Force}]}{[\text{Area}]}\cdot\frac{[\text{Length}]}{[\text{Length}]} = \frac{[\text{Energy}]}{[\text{Volume}]} = [\text{Energy}]^4$$

another example . . .

A quantum mechanics text gives the Bohr radius as $a_0 = \frac{\hbar^2}{m_e e^2}$

But I see this as . . .

$$a_0 = \frac{1}{m_e e^2} = \frac{\hbar c}{e^2} \cdot \frac{\hbar c}{m_e} = (137.036) \cdot \frac{197.33\,\text{Mev fm}}{0.511\,\text{MeV}} = 0.52918\,\text{Å}$$

or whatever units you prefer . . .

$$\hbar c \approx 1.9733 \times 10^{-5}\ \text{eV cm}$$
$$c \approx 2.9979 \times 10^{10}\ \text{cm s}^{-1}$$

or maybe even . . .

$$\hbar c \approx 1.9733 \times 10^{3}\,\text{eV Å}$$
$$c \approx 2.9979 \times 10^{18}\ \text{Å s}^{-1}$$

OK, why not use ***ergs*** or ***Joules*** and ***centimeters*** or ***meters***
You can if you want but . . .

better to be like **Hans Beth** and use units scaled to the problem at hand

size of a nucleon/nucleus ~ 1 fm
energy levels in a nucleus ~ 1 MeV

$$\text{atomic/molecular sizes} \sim \text{Å}$$
$$\text{atomic/molecular energies} \sim \text{eV}$$

supernova explosion energy $1\ \text{Bethe} \equiv 10^{51}\ \text{erg}$

electric charge and potentials/energies

one elementary charge $1\ e \approx 1.6022 \times 10^{-19}\ \text{Coulombs}$

One Coulomb falling through a potential difference of 1 Volt = 1 Joule= 10^7 erg

$$1\ \text{eV} \approx 1.6022 \times 10^{-19}\ \text{J} = 1.6022 \times 10^{-12}\ \text{erg}$$

or

$$1\ \text{MeV} \approx 1.6022 \times 10^{-6}\ \text{erg}$$
$$1\ \text{erg} \approx 6.241 \times 10^{5}\ \text{MeV} \sim 10^{6}\,\text{MeV}$$

fine structure constant $\alpha_{\rm em}$

SI $\quad \dfrac{e^2}{4\pi\epsilon_0\ \hbar c} \approx \dfrac{1}{137.036}$

cgs $\quad \dfrac{e^2}{\hbar c} \approx \dfrac{1}{137.036}$

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particle masses, atomic dimensions, *etc.*

electron rest mass $m_{\rm e} \approx 0.511\,{\rm MeV}$

proton rest mass $m_{\rm p} \approx 938.26\,{\rm MeV}$

neutron-proton mass difference $m_{\rm n} - m_{\rm p} \approx 1.293\,{\rm MeV}$

atomic mass unit $1\ {\rm amu} \approx 931.494\,{\rm MeV}$

Avogadro's number $N_{\rm A} \approx 6.022 \times 10^{23}\,\dfrac{\rm amu}{\rm g}$

Handy Facts: Solar System

solar mass $M_\odot \approx 1.989 \times 10^{33}\,{\rm g} \approx 10^{60}\,{\rm MeV}$
solar radius $R_\odot \approx 6.9598 \times 10^{10}\,{\rm cm}$
solar luminosity $L_\odot \approx 3.9 \times 10^{33}\,{\rm erg\,s^{-1}}$

$1\,{\rm A.U.} \approx 1.4960 \times 10^{13}\,{\rm cm}$ radius of earth's orbit around sun

earth mass $M_{\rm earth} \approx 3 \times 10^{-6}\,M_\odot$ $\quad M_{\rm Jupiter} \sim 300\,M_{\rm earth} \sim 10^{-3}\,{\rm M}_\odot$
earth radius $R_{\rm earth} \approx 6.3782 \times 10^{8}\,{\rm cm} \sim 10^{-2}\,R_\odot$
Jupiter orbital radius $\sim 5\,{\rm A.U.}$
solar system diameter $\sim 100\,{\rm A.U.}$

sidereal day $\approx 8.6164091 \times 10^{4}\,{\rm s} \sim 10^{5}\,{\rm s}$
sidereal year $\approx 3.1558 \times 10^{7}\,{\rm s} \sim \pi \times 10^{7}\,{\rm s} \sim 3 \times 10^{7}\,{\rm s}$
1 dog year $\approx 7.0000\,{\rm yr}$

We can do all this for spacetime too !

Define the Planck Mass $m_{\rm pl} \equiv \left(\dfrac{\hbar c}{\rm G}\right)^{1/2}$

$$m_{\rm pl} \approx 1.2211 \times 10^{22}\,{\rm MeV} \sim 10^{22}\,{\rm MeV}$$

. . . and now the Gravitational constant is just . ${\rm G} = \dfrac{1}{m_{\rm pl}^2}$

The essence of General Relativity:

There is no gravitation: in *locally* inertial coordinate systems, which the Equivalence Principle guarantees are always there, the effects of gravitation are absent!

The Einstein Field equations have as there solutions global coordinate systems which cover big patches of spacetime

A convenient coordinate system for *weak* & *static* (no time dependence) gravitational fields is given by the following coordinate system/metric:

$$ds^2 = -(1+2\varphi)dt^2 + (1-2\varphi)\left(dx^2+dy^2+dz^2\right)$$

This would be a decent description of the spacetime geometry and gravitational effects around the earth, the sun, and white dwarf stars, but not near the surfaces of neutron stars.

It turns out that in a weak gravitational field the time-time component of the metric is related to the Newtonian gravitational potential by . . .

$$g_{00} \approx -1 - 2\varphi$$

Where the Newtonian gravitational potential is $\varphi \approx -\frac{G\,M}{R}$

$$G \equiv \frac{1}{m_{\rm pl}^2}$$

$$m_{\rm pl} \approx 1.221 \times 10^{22}\,{\rm MeV}$$

$$\hbar\,c \approx 197.33\,{\rm MeV\,fm}$$

$$1\,{\rm fm} = 10^{-13}\,{\rm cm}$$

$$1\,{\rm MeV} \approx 1.6022 \times 10^{-13}\,{\rm Joules}$$

$$\varphi \approx -\frac{M\,\hbar\,c}{m_{\rm pl}^2\,R}$$

dimensionless !

Characteristic Metric Deviation

OBJECT	MASS (solar masses)	RADIUS (cm)	Newtonian Gravitational Potential
earth	3×10^{-6}	6.4×10^{8}	$\sim 10^{-9}$
sun	1	6.9×10^{10}	$\sim 10^{-6}$
white dwarf	~1	5×10^{8}	$\sim 10^{-4}$
neutron star	~1	10^{6}	~0.1 to 0.2

Handy Facts: the Universe

$$1 \text{ parsec (pc)} \approx 3.2615 \text{ light year (l.y.)}$$
$$1 \text{ mega-parsec (Mpc)} \approx 3.0856 \times 10^{24} \text{ cm} \ \sim 3 \times 10^{24} \text{ cm}$$

size galaxy $\sim 1\,\mathrm{Mpc}$, $10^{12}\,\mathrm{M}_\odot$ dark matter
size galaxy (visible/baryons) $\sim 100\,\mathrm{kpc}$, $10^{11}\,\mathrm{M}_\odot$ baryons
galaxy density (inside causal horizon) $\sim 1\,\mathrm{Mpc}^{-3}$

big galaxy clusters ~ 1000 galaxies
Virgo Cluster distance $\sim 16\,\mathrm{Mpc}$
Coma Cluster distance $\sim 55\,\mathrm{Mpc}$

$$\mathrm{H}_0 = \text{expansion rate of universe (current epoch)} = (100\ \mathrm{h})\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$$
$$\mathrm{h} \approx 0.71 \quad e.g.,\ \mathrm{WMAP3}$$

$$\mathrm{H}_0 \approx \left(2.13 \times 10^{-39}\,\mathrm{MeV}\right)\mathrm{h} \approx \left(3.24 \times 10^{-18}\,\mathrm{s}^{-1}\right)\mathrm{h}$$

$$\mathrm{H}_0^{-1} \approx (9.78\,\mathrm{Gyr})\,\mathrm{h}^{-1} \qquad \text{age of universe} \approx 13.7\,\mathrm{Gyr}$$

$$\Omega \equiv \text{closure fraction} = \frac{\rho}{\rho_c} \qquad \Omega_{\text{Dark Matter}} \approx 0.23 \qquad \Omega_{\text{vac}} \approx 0.73 \qquad \Omega_{\text{baryon}} \approx 0.04$$

$$\rho_c = \text{closure density} = \frac{3\mathrm{H}_0^2}{8\pi \mathrm{G}} = \frac{3}{8\pi}\mathrm{H}_0^2\, m_{\text{pl}}^2 \approx \left(8.1 \times 10^{-35}\,\mathrm{MeV}^4\right)\ \mathrm{h}^2$$
$$\approx \left(1.054 \times 10^4\,\mathrm{eV\,cm^{-3}}\right)\ \mathrm{h}^2 \sim 10^{-5}\,\frac{\mathrm{amu}}{\mathrm{cm}^3}$$

$$\rho_{\text{vac}} = \text{dark energy density} \approx \left(3.9\,\frac{\mathrm{keV}}{\mathrm{cm}^3}\right)\cdot\left(\frac{\mathrm{h}}{0.71}\right)^2\cdot\left(\frac{\Omega_{\text{vac}}}{0.73}\right) \sim 4\,\mathrm{keV\,cm^{-3}}$$

Rates and Cross Sections

$$[\text{Rate}] = [\text{Flux}] \cdot [\text{Cross Section}]$$

$$\left[\mathrm{s}^{-1}\right] = \left[\frac{\#}{\mathrm{cm}^2\ \mathrm{s}}\right] \cdot \left[\mathrm{cm}^2\right]$$

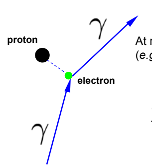
Eddington Luminosity

Sir Arthur Eddington
www.sil.si.edu

Photon scattering-induced momentum transfer rate to electrons/protons must be less than gravitational force on proton

$$\gamma + e \rightarrow e + \gamma$$

Electrons tied to protons via Coulomb force



At radius r where interior mass is $M(r)$ and photon energy luminosity ($e.g.$, in ergs s^{-1}) is $L_\gamma(r)$ the forces are equal when

$$\frac{L_\gamma(r)}{4\pi r^2 c}\cdot \sigma_T = \frac{\mathrm{G}\ M(r)\ m_p}{r^2}$$

Flux of photon momentum

Gravitational force on proton with mass m_p

Thomson cross section $\sigma_T \approx 6.65 \times 10^{-25}\,\mathrm{cm}^2 \sim 10^{-24}\,\mathrm{cm}^2 = 1\,\mathrm{barn}$

$$\Rightarrow L_\gamma^{\text{Eddington}} = \frac{4\pi \mathrm{G}\, M\, m_p c}{\sigma_T} = 4\pi\left[\frac{M\, m_p}{m_{pl}^2}\right]\frac{(\hbar c)c}{\sigma_T}$$
$$\approx 10^{38}\,\mathrm{erg\,s^{-1}}\left[\frac{M}{\mathrm{M}_\odot}\right]$$