

Boundary Layers and Basic Fluids

In this lecture, we will look at a few basic fluids problems, using the boundary layer as a launching point.

The Boundary Layer

Boundary layers play an important role in aerodynamics, and also in thermal transport. The basic idea is that surfaces impose a no-slip condition on the fluid around it, so that when exposed to an external flow there is a sheath of stagnant air clinging to the surface. This is why pollen on a car is not completely blown off by driving through the medium we call air. It will be our happy job to estimate the thickness of a boundary layer.

First, we should ponder what basic property of a fluid gives rise to the “clingy” boundary layer. Is it density, thermal conductivity, heat capacity, molecular weight, electrical conductivity? Hopefully none of those sounded right to you. It’s viscosity, of course. That’s what makes a fluid “sticky,” or even “wet,” one might say. So let’s Buckingham Pi this problem.

We’re after the boundary layer thickness, δ , and know we need the viscosity, ν (we’ll choose the dimensionally simpler kinematic viscosity). The boundary layer is likely not the same along a surface, so we’ll include a length scale along the flow, x . Presumably we need velocity (pollen on your car less likely to withstand a high-speed chase), and we may find density to be relevant. So our table:

i	v_i	units	notes
1	δ	m	sought quantity
2	x	m	length along flow
3	v	m/s	fluid velocity
4	ν	m ² /s	kinematic viscosity
5	ρ	kg/m ³	density

In looking for dimensionless combinations, we find that kg only appears in one of the variables. Obviously we cannot combine any other variables to cancel kg. So unless we can think of some other relevant parameter with kg (and I can’t), we have no choice but to conclude that density of the medium is not relevant to the problem! This highlights another perk elucidated by Buckingham Pi: because any physics equation *can* be represented in dimensionless form, any combination where only a single variable contains a given fundamental unit must not be relevant to the problem.

So eliminating ρ on these grounds, we have $n = 4$ and $r = 2$ so that we look for $n - r = 2$ dimensionless Π variables. Easy:

$$\Pi_1 = \frac{\delta}{x}, \Pi_2 = \frac{xv}{\nu}.$$

We recognize the second one as the now-familiar Reynolds number. We make the usual formulation:

$$\Pi_1 = f(\Pi_2), \rightarrow \delta = xf\left(\frac{xv}{\nu}\right).$$

And now comes the physics intuition. I would expect the boundary layer to become more significant the higher the viscosity. So let’s try $f(x) = x^{-1}$. This makes $\delta = \nu/v$. Looks simple. Do we like it? First, notice that it eliminates x as a consideration. How do we feel about that? But let’s look quantitatively. For a car on a residential street, with $v = 15$ m/s, we would get $\delta \approx 10^{-6}$ m, or a micron (recall $\nu = 1.5 \times 10^{-5}$ m²/s for air). This does not sit well. Goodbye pollen. We want viscosity in the numerator, but not so powerful as it is here. We can try $f(x) = x^{-\frac{1}{2}}$ as a moderated scheme. Now,

$$\delta = x\left(\frac{\nu}{xv}\right)^{\frac{1}{2}} = \sqrt{\frac{x\nu}{v}}. \quad (1)$$

We can tell by the fact that this is set apart on its own line that it must be right. Note that x is now a participant. Quantitatively, the situation from before, with $x = 1$ m as a characteristic scale, results in 1 mm. This is a much better match to intuition. We'll see in the next section why we should have more confidence in this answer.

For now, let's flesh out what this means. The first thing to notice is that the boundary layer grows (slowly) as one moves down the flow. We can imagine this as sort-of a parabolic bow-wave-looking thing. Next, no matter what familiar situations we plug in on a human scale, we do not stray far from the millimeter scale for the boundary layer. A jet aircraft might see 0.2 mm on its wing. Kinematic viscosity in water is lower than that in air, but so are velocities, typically, so that we end up in the same ballpark.

When we get to convective heat flow, we will see that this is reasonably treated as conduction across the static boundary layer.

Fluids for Adults

I do not mean this in the usual cultural sense, but in the sense that when we grow up and get serious about describing fluid flow, we don't get far without embracing the Navier-Stokes equation for an incompressible, viscous fluid. It reads thusly:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \frac{1}{\rho} \nabla p + \mathbf{g}, \quad (2)$$

where \mathbf{u} is the vector velocity of the fluid, ν is kinematic viscosity, ρ is fluid density, p is pressure, and \mathbf{g} represents an external acceleration, like gravity. The left hand side describes inertial flow of the fluid, while the right covers the forces acting on the fluid. Broken down term by term, we describe each as:

- simple vector acceleration
- “convection” which really describes directional changes in the fluid
- viscosity, which looks like diffusion of velocity
- pressure gradient: what's pushing on the fluid
- external acceleration

If we release a chunk of air or water on the moon, we only keep the first and last terms, and see this as the usual reaction to gravity.

Boundary Layer from Navier-Stokes

Let's look at Eq. 2 with the eyes of a hobo¹. By this, I mean that we will use hobo derivatives. For a boundary layer problem, we imagine a fluid flow encountering a plate parallel to the flow. We want to know what the boundary layer thickness is some distance x from the leading edge. We imagine steady state, so can drop the first term. Ignore gravity or other external accelerations. Pressure gradients are not at play. So we are left with

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u}.$$

Okay, so the term on the left. We have oncoming fluid at velocity v , and at the boundary layer height we're interested in it must transition to zero velocity over the length scale x . So the change in velocity over the change in distance has the scale v/x and the term on the left is v^2/x . That's how we do hobo derivatives. How about the term on the right? Going vertically from the surface, we see the velocity field changes from zero to v across the boundary layer thickness δ . So the hobo form of the Laplacian term is $\nu v/\delta^2$. Equating these two, we find that $\delta^2 = x\nu/v$, which is wholly consistent with what we found in Eq. 1.

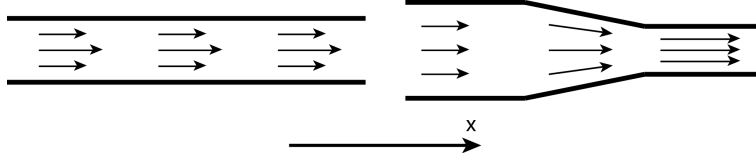


Figure 1: Example pipe flow velocity fields

Interpretation of the Convective Term

Imagine the two flows pictured in Figure 1 for flows through a pipe oriented along the x -direction. Let's evaluate the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ at various places in each pipe. In Cartesian coordinates, the term breaks up to look like

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) \mathbf{u}. \quad (3)$$

In the straight pipe, both u_y and u_z are zero, so only the first term in Eq. 3 lives. But since the flow is steady along the pipe, the derivative along x is zero and the whole term vanishes. In the reduction case, Eq. 3 likewise vanishes in the straight portions of the pipe. But in the middle part, all components of \mathbf{u} are live, and the velocity is changing as a function of position (speeding up even on the centerline, where u_y and u_z are zero).

If we had a steady shear flow all in one direction, (say $\mathbf{u} = \langle u(y), 0, 0 \rangle$), we still get zero convection. Even though the flow changes with position (y -coordinate), the x -component is not changing along the x -direction. So we might say that the convective term tells us if fluid elements are changing direction or changing speed as a function of position in the flow. Note that this is practically the definition of acceleration, but as a function of position rather than time. The pattern may be static with respect to the coordinate system, but fluid elements actually do change velocity (accelerate) as they move ("vect") along the flow. Thus the term convection.

Pipe Flow

Let's apply the Navier-Stokes machinery to steady laminar flow in a circular pipe of radius, R . We will use cylindrical coordinates, so that flow is along the z -direction. To fight viscosity, we need a pressure gradient along the pipe. Because the flow is static, and all velocities are confined to be along the pipe axis, both the time derivative and convective terms are zero. We are looking for the velocity profile as a function of radius for a given pressure gradient, which will allow us to calculate the total flow rate. The truncated equation is then

$$\nu \nabla^2 \mathbf{u} = \frac{1}{\rho} \nabla p.$$

The velocity is strictly along the z -axis, so the pressure gradient can only be in this axis. We can therefore simplify to

$$\rho \nu \nabla^2 u_z = \frac{\partial p}{\partial z} \equiv -\alpha,$$

where we have adopted α to represent the pressure gradient, for notational convenience (negative sign because pressure must be lower at more positive z values based on the forced flow direction). The Laplacian operator contains second derivatives with respect to z , r , and θ . For our laminar axial flow, only the derivative with respect to r is non-zero (no change along z , no azimuthal asymmetry). The Laplacian in cylindrical coordinates turns this into

$$\rho \nu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) = \rho \nu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) = -\alpha.$$

¹A hobo is an American icon from days past. Imagine today's homeless wandering the country on freight trains with sacks on sticks over their shoulders carrying their goods. Hobos are practical opportunists, and anything but formal.

A solution to this equation has the form $u_z(r) = Ar^2 + Br + C$, which turns into $(2A + 2A + B/r) = -\alpha/\rho\nu$. Since the right hand side is a constant, we cannot tolerate a function of r on the left, so $B = 0$ and $A = -\alpha/4\rho\nu$. The no-slip condition requires that $u_z(R) = 0$, which forces $C = \alpha R^2/4\rho\nu$, which gets put together into the formula describing what is known as Poiseuille Flow:

$$u_z(r) = \frac{\alpha}{4\rho\nu}(R^2 - r^2).$$

So the flow is quadratic, becoming zero at the walls (as mandated) and maximum at the center. Note that the second derivative (Laplacian) is constant, in accordance with a constant pressure gradient. The flow rate, in volume per second is the integral of velocity over the area:

$$F = 2\pi \int_0^R r dr u_z(r) = \frac{\alpha\pi R^4}{8\rho\nu} = \frac{\alpha AR^2}{8\rho\nu}.$$

This indicates that flow rate is *strongly* sensitive to pipe radius for a given pressure, or that pressure must rise dramatically to sustain the same flow rate in a smaller diameter pipe. The final form is because the average flow velocity (area-weighted) in the pipe is F/A , so that $v_{\text{avg}} = \alpha R^2/8\rho\nu$.

Putting some numbers to it, a microfluidics experiment (small means laminar and not turbulent) has a 1 mm diameter tube that is 1 m long, with pressure generated from a reservoir whose surface is 0.1 m above the (horizontal) tube. The pressure gradient, $\alpha = \rho gh/L \approx 1000 \text{ Pa/m}$. This produces a flow of $2 \times 10^{-8} \text{ m}^3/\text{s}$, which seems like a tiny number. Let's say it drips out at the end, in drops 2 mm in radius—translating to $3 \times 10^{-8} \text{ m}^3$. So it's approximately a drop per second. The average velocity of fluid in the tube is 30 mm/s. A free-stream governed only by gravity (no viscosity) would result in $v = \sqrt{2gh} \approx 1.4 \text{ m/s}$, so friction in the tube is clearly a major influence on the flow.

Poiseuille flow is valid in the laminar regime, which when confined to a pipe goes up to $\text{Re} \approx 2000$. By the time we reach $\text{Re} > 4000$ we find fully-developed turbulence. Our example has $Rv/\nu \sim 10^{-5}/10^{-6} = 10$, which is very safely laminar. We would have to go up to $R \approx 2.5 \text{ mm}$ with an average velocity of 0.8 m/s for the same $h = 0.1$ setup, which would result in (600 times the flow).

Surface Tension

This may not be a natural follow-on to our fluids treatment (will pick up turbulence, waves later). But hey—I wanted to stick it somewhere. Surface tension for water is $\gamma \approx 0.07 \text{ N/m}$ at room temperature. It's a strange unit somewhere between pressure and force, but may be usefully thought of as energy per unit area. This property governs raindrop size, capillary action, water skater bugs, an over-full cup that does not spill, etc.

Think of a spherical drop of water, and now imagine adding more volume to the drop so that it expands at the equator, somehow. The total force opposing the stretch is γ times the circumference: $F = 2\pi R\gamma$. You might say this is the force holding the two hemispheres together. In energy terms, if we add a “belt” around the equator of width w , the area added is $2\pi R w$. This requires energy $2\pi R w \gamma$. Since energy is force times distance, $F w$, the force is $2\pi R \gamma$ —which is the same result as before.

Example: Overfull Cup

Let's do an example problem: how much can we over-fill a cup, relying on surface tension to hold it together? The central idea is that surface tension resists adding new area, unless the gravitational potential energy to be traded provides sufficient compensation. Even though water wets well to glass, sending out a finger to creep over the cup edge costs a lot in surface tension without providing much potential energy relief from the overburden, so this does not happen.

We will model our cup as having approximate radius, R , and height, h , arranged so that the curve at the edge has radius, h to make a 90° arc—as depicted in Figure 2. For mathematical convenience, we make R the



Figure 2: Geometry for overflow atop the cup. We investigate what keeps the bulge from pushing out to the side.

radius of the flat part, but it is approximately the radius of the cup, since $h \ll R$. In terms of surface area, the flat top part is just πR^2 . We approximate the curved rim as having average radius $R + \frac{h}{2}$, so that the area is the circumference times the arclength: $2\pi(R + \frac{h}{2})\frac{\pi}{2}h$. The ratio of these areas (curved ring boundary to flat top) is $\pi(1 + \frac{h}{2R})\frac{h}{R} \sim \pi h/R$, which means that if the bulge feature tries to relax, it effectively takes material off the top and deposits it around the sides. The thickness ratio (reduced height of bulge makes the feature wider) follows the ratio of areas, so that a Δh translates to a $\Delta R \approx \frac{R}{\pi h} \Delta h$.

In spreading out the bulge, we add area to the top disk in the amount $\Delta A = 2\pi R \Delta R = 2R^2 \Delta h/h$, requiring energy (from surface tension) $\Delta E_{S.T.} = \gamma \Delta A = 2\gamma R^2 \Delta h/h$. Meanwhile, the water we took off the top has volume $\pi R^2 \Delta h$ and we move it down roughly $h/2$ to distribute it on the outer ring. This translates to a potential energy change of $mgh/2 = \frac{1}{2}\pi\rho R^2 hg \Delta h$. The bulge is stable if the energy required to change the area is greater than the potential energy loss. So the ratio of energies is useful, and is $4\gamma/\pi\rho gh^2$. Note that the cup diameter, R , is gone. When h is very small, this ratio is large, meaning that it costs more in surface energy than is gained in potential. The ratio becomes unity when

$$h = \sqrt{\frac{4\gamma}{\pi\rho g}},$$

which we compute to be 3 mm for water. Go try it. I found that it works!

Intuitively, for a given cup size, reducing the height by some Δh and putting that material onto the outer ring results in a large radius increase when h is small: not much width to the belt on which we pile that water. Thus the surface area on the top of the disk increases a lot at great cost via surface tension. And at the same time, there is not much height change so little savings in potential. For a thicker layer, there's more space to accommodate the displaced volume, so the radius does not increase much (not much surface tension cost) while the potential energy gain increases due to a larger vertical displacement. As h grows, the surface energy cost for pushing out decreases, while the potential energy gain increases. Both factors at once is why the ratio has h^2 , which is why the calculation of h goes as the square root of the relevant pieces.

The Cheerio Effect

Did you ever amuse yourself by separating floating cereal units only to find them bind back together in a flotilla? This is a surface tension effect related to capillary action: liquid climbs up the sides of the tasty units, much as it does the sides of a container. There is some potential energy cost in doing so. By joining side-by-side, two morsels can share the potential energy burden, so it's more efficient: less total volume is hoisted above the main surface.